Math 8: Calculus in one and several variables
Spring 2017 - Homework 8

Return date: Wednesday 05/24/17

---

keywords: *directional derivatives, gradients, extreme values*

exercise 1. *(3 points)* Let $f(x, y)$ be the function $f(x, y) = x^2 \cdot \ln(y)$.

a) Find the gradient $\mathbf{grad}(f)$ of f.
Solution: The gradient is the vector of the partial derivatives.

$$\mathbf{grad}(f) = \langle f_x, f_y \rangle = \langle 2x \ln(y), \frac{x^2}{y} \rangle.$$

b) Evaluate the gradient at the point $P = (3, 1)$.
Solution: At $(x, y) = (3, 1)$ we get

$$\mathbf{grad}(f)(3, 1) = \langle f_x(3, 1), f_y(3, 1) \rangle = \langle 2 \cdot 3 \ln(1), \frac{3^2}{1} \rangle = \langle 0, 9 \rangle.$$

c) Find the rate of change $D_{\mathbf{u}}f(P)$ of f at P in direction from P towards the point $Q = (-2, 13)$.
Solution: We first have to find the direction vector $\mathbf{u}$ of unit length. For the direction $\tilde{\mathbf{u}}$ we get

$$\tilde{\mathbf{u}} = \overrightarrow{PQ} = Q - P = \langle -5, 12 \rangle. \text{ Then } \mathbf{u} = \frac{\tilde{\mathbf{u}}}{|\tilde{\mathbf{u}}|} = \left\langle \frac{-5}{13}, \frac{12}{13} \right\rangle = \langle a, b \rangle.$$

Then by Ch. 14.6, **Theorem 3** of the book the rate of change $D_{\mathbf{u}}f(P)$ is

$$D_{\mathbf{u}}f(3, 1) = f_x(3, 1) \cdot a + f_x(3, 1) \cdot b = 0 \cdot \left(\frac{-5}{13} \right) + 9 \cdot \left(\frac{12}{13} \right) = \frac{108}{13} \simeq 8.03.$$

Hence the rate of change or slope of f at P in direction $\mathbf{u}$ is about 8.03.

exercise 2. *(2 points)* Find the maximal rate of change of the function f at the given point P and the direction in which it occurs for

a) $f(x, y) = \cos(xy)$ at $P = (0, 1)$.
Solution: The gradient vector at the point P indicates the direction of the maximal increase or rate of change. We have

$$\mathbf{grad}(f) = \langle f_x, f_y \rangle = \langle -y \sin(xy), -x \sin(xy) \rangle. \text{ Hence } \mathbf{grad}(f)(0, 1) = \langle 0, 0 \rangle.$$

This means that we are at a critical point so the directional derivative in all directions is zero. Hence the maximal rate of change at $P = (0, 1)$ is equal to zero.

---

b) $f(x,y,z) = x\ln(yz)$. at $P = (1,2,\frac{1}{2})$.
Solution: In this case we have

$$\mathbf{grad}(f) = \langle f_x, f_y, f_z \rangle = \left\langle \ln(yz), \frac{x}{y}, \frac{x}{z} \right\rangle. \text{ Hence } \mathbf{grad}(f)(1,2,1/2) = \langle 0, 1/2, 2 \rangle.$$

Then the direction of maximal increase is $\mathbf{grad}(f)(1,2,\frac{1}{2}) = \langle 0, \frac{1}{2}, 2 \rangle$.
Scaling this vector to unit length we get

$$\mathbf{u} = \frac{\langle 0, \frac{1}{2}, 2 \rangle}{|\langle 0, \frac{1}{2}, 2 \rangle|} = \frac{2}{\sqrt{17}} \cdot \langle 0,\, 1/2\,, 2 \rangle.$$

Then by Chapter 14.6, **Theorem 15** of the book, the maximal rate of change of f at P is

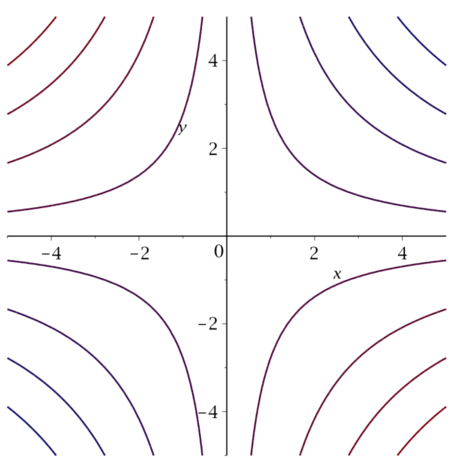
$$D_{\mathbf{u}} f(1,2,1/2) = |\mathbf{grad}(f)(1,2,1/2)| = \frac{\sqrt{17}}{2}.$$

exercise 3. *(4 points)* Let $f(x,y) = xy$.

a) Find the level sets $f(x,y) = k$ for $k = -1, 0$ and 1 and sketch them in the xy-plane.
Solution: For the level sets $L_f(k)$ of f we get $k = z = f(x,y) = xy$, hence

$$\begin{aligned} &\text{Case 1.) } \underline{k = 0} \quad : \quad 0 = xy \Leftrightarrow x = 0 \;\; \text{(y-axis)} \quad \text{or} \quad y = 0 \;\; \text{(x-axis)}. \\ &\text{Case 2.) } \underline{k \neq 0} \quad : \quad k = xy \Leftrightarrow y = \frac{k}{x} \quad \text{(hyperbola)}. \end{aligned}$$

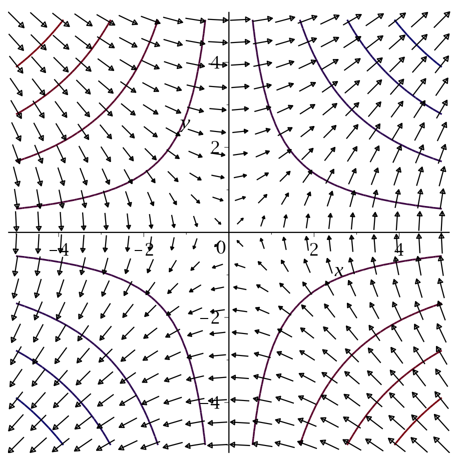
For $k \neq 0$ the level sets are hyperbolas in the xy-plane. A picture is shown below:



b) Find the gradient $\mathbf{grad}(f)$ of f.
Solution: We get

$$\mathbf{grad}(f) = \langle f_x, f_y \rangle = \langle y, x \rangle.$$

c) Sketch the gradient vector $\mathbf{grad}(f)(x, y)$ at several points along each of the level curves in part a). Be sure your sketch shows clearly the direction in which the gradient points and its relationship to the level set.
Solution: The gradient vector $\mathbf{grad}(f)(x, y)$ at $P = (x, y)$ is perpendicular to the level set passing through this point. It indicates the direction of maximal increase.



d) Find the tangent line to the level set $f(x, y) = -1$ at the point $P = (-2, \frac{1}{2})$.
Solution: At the point $P = (-2, \frac{1}{2})$ the gradient is $\mathbf{grad}(f)(-2, \frac{1}{2}) = \langle \frac{1}{2}, -2 \rangle$.
Then gradient vector is perpendicular to the level set at this point, so the direction vector $\mathbf{v}$ of the tangent line satisfies

$$\mathbf{v} \perp \mathbf{grad}(f)(-2, 1/2) \text{ hence } \mathbf{v} \bullet \langle \frac{1}{2}, -2 \rangle = 0. \text{ We can take } \mathbf{v} = \langle 4, 1 \rangle.$$

Then the tangent line L is

$$L : P + t \cdot \mathbf{v} = \langle -2 + 4t, 1/2 + t \rangle \text{ for } t \text{ in } \mathbb{R}.$$

exercise 4. *(4 points)* Find the local maxima, minima, and saddle points of the following functions.

a) $f(x, y) = 2 - x^4 + 2x^2 - y^2$.
Solution: We first find the critical points where $\mathbf{grad}(f) = \langle 0, 0 \rangle$. Then we determine

the nature of the critical point using the Second Derivatives (SD) Test (see book page 1001):

$$\mathbf{grad}(f) = \langle -4x^3 + 4x, -2y \rangle = \langle 0, 0 \rangle \Leftrightarrow x = 0 \text{ or } x = 1 \text{ or } x = -1 \text{ and } y = 0.$$

The critical points are $P = (-1, 0), Q = (0, 0)$ and $R = (1, 0)$.
To apply the SD test we have to find the second order derivatives of f:

$$f_{xx} = -12x^2 + 4 \ , \ f_{yy} = -2 \text{ and } f_{xy} = f_{yx} = 0.$$

For a critical point (a, b) we have to apply the SD test. Therefore we have to calculate

$$D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - (f_{xy}(a, b))^2 \text{ and } f_{xx}(a, b).$$

We get

P : $D(-1, 0) = (-8)(-2) - 0^2 = 16 > 0$ and $f_{xx} = -8 < 0 : f(P) = 3$ is a local maximum.
Q : $D(0, 0) = (4)(-2) - 0^2 = -8 < 0 : f$ has a saddle point in Q.
R : $D(1, 0) = (-8)(-2) - 0^2 = 16 > 0$ and $f_{xx} = -8 < 0 : f(R) = 3$ is a local maximum.

b) $f(x, y) = (x^2 + y^2)e^x$.
Solution: Following the steps from part a) we get, as $e^x > 0$:

$$\mathbf{grad}(f) = \langle (2x + x^2 + y^2)e^x, 2ye^x \rangle = \langle 0, 0 \rangle \overset{e^x > 0}{\Leftrightarrow} 2x + x^2 + y^2 = 0 \text{ and } y = 0.$$

To meet both conditions y has to be zero. That simplifies the first condition to $2x + x^2 = 0$ or ($x = 0$ or $x = -2$).
Hence the critical points are $P = (-2, 0)$ and $Q = (0, 0)$.
To apply the SD test we have to find the second order derivatives of f:

$$f_{xx} = (2 + 4x + x^2 + y^2)e^x \ , \ f_{yy} = 2e^x \text{ and } f_{xy} = f_{yx} = 2ye^x.$$

For a critical point (a, b) we have to apply the SD test. Therefore we have to calculate

$$D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - (f_{xy}(a, b))^2 \text{ and } f_{xx}(a, b).$$

We get

P : $D(-2, 0) = (-2e^{-2}) \cdot 2e^{-2} - 0^2 = -4e^{-4} < 0 : f$ has a saddle point in P.
Q : $D(0, 0) = (2e^0)(2e^0) - 0^2 = 4 > 0$ and $f_{xx} = 2 > 0 : f(Q) = 0$ is a local minimum.

Math 8: Calculus in one and several variables
Spring 2017 - Homework 8

Return date: Wednesday 05/24/17

---

exercise 5. *(4 points)* Let D be the closed triangular region with vertices

$$A = (0,0)\ ,\ B = (0,2)\ \text{ and }\ C = (4,0).$$

Find the absolute maximum and minimum of the function

$$f(x,y) = x + y - xy \ \text{ in the region } D.$$

Solution: We have to find all possible extrema of the function in the region D. The possible extrema in the interior of D are the critical points. These satisfy $\mathbf{grad}(f)(a,b) = \langle 0,0\rangle$. The other possible extrema are the extreme values of f on the boundary ∂D of D. We first check for critical points in the interior:

$$\mathbf{grad}(f) = \langle f_x, f_y\rangle = \langle 1-y, 1-x\rangle = \langle 0,0\rangle \Leftrightarrow x = y = 1 \ \text{ and }\ f(1,1) = 1.$$

Hence the only critical point is $P = (1,1)$ with $f(1,1) = 1$.
We now look at the boundary ∂D. We can parametrize the three boundary segments L_1, L_2 and L_3 by

$$L_1(t) = \langle 0, 2t\rangle\ ,\ L_2(t) = \langle 4t, 0\rangle \ \text{ and }\ L_3(t) = \langle 4t, 2-2t\rangle, \quad \text{where}\ \ t \in [0,1].$$

Using this parametrization we can investigate the function on the boundary. We get:

$$f(L_1(t)) = 2t\ ,\ f(L_2(t)) = 4t \ \text{ and }\ f(L_3(t)) = 4t+(2-2t)-4t(2-2t) = 8t^2-6t+2, \quad \text{where}\ \ t \in [0,1].$$

Now $f(L_1(t))$ and $f(L_2(t))$ are linear functions of t. In the interval $t \in [0,1]$ the maxima and minima of these functions are attained on the boundary $t = 0$ or $t = 1$. We get the possible extrema:

$$f(L_1(t)) : A = (0,0),\ f(0,0) = 0\ (t=0) \ \text{ and }\ B = (0,2),\ f(0,2) = 2\ (t=1)$$
$$f(L_2(t)) : A = (0,0),\ f(0,0) = 0\ (t=0) \ \text{ and }\ C = (4,0),\ f(4,0) = 4\ (t=1).$$

The function $f(L_3(t))$ might have an extremum in the interior of the interval $[0,1]$.
We get with $f(L_3(t)) = 8t^2 - 6t + 2$:

$$\frac{\partial}{\partial t} f(L_3(t)) = 0 \Leftrightarrow 16t - 6 = 0 \ \text{ hence }\ t = \frac{3}{8}.\ \text{So}\ \ P = L_3(3/8) = \left(\frac{3}{2}, \frac{5}{4}\right),\ f(L_3(3/8)) = \frac{7}{8}.$$

Comparing the values for the possible extrema, we get:

Minimum of $f : 0 = f(0,0)$ at A. Maximum of $f : 4 = f(4,0)$ at C.

Math 8: Calculus in one and several variables
Spring 2017 - Homework 8

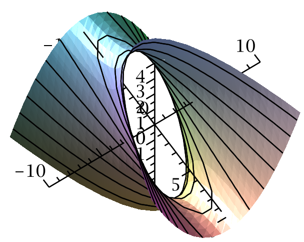
Return date: Wednesday 05/24/17

---

exercise 6. *(3 points)* Find the points on the surface

$$S: \; y^2 = 9 + xz \quad \textit{(see below)}$$

that are closest to the origin.
Hint: Instead of minimizing the distance from the origin, you can also minimize the square of the distance.



Solution: For any point (x, y, z) in S the square of the distance d to the origin is

$$d((x, y, z), (0, 0, 0))^2 = \tilde{d} = |(x, y, z) - (0, 0, 0)|^2 = x^2 + y^2 + z^2.$$

We know furthermore that the surface is determined by the equation $9 + xz = y^2 \geq 0$. We can substitute this into the equation for the distance.

$$\tilde{d} = x^2 + (9 + xz) + z^2.$$

This function depends only on x and z. The critical points of this function are the points, where the gradient is equal to $\langle 0, 0\rangle$. We get

$$\mathbf{grad}(\tilde{d}) = \langle \tilde{d}_x, \tilde{d}_z\rangle = \langle 2x + z, x + 2z\rangle = \langle 0, 0\rangle \Leftrightarrow x = 0 \;\text{ and }\; z = 0.$$

For the second order derivatives we get:

$$\tilde{d}_{xx} = 2 \;,\; \tilde{d}_{xz} = \tilde{d}_{zx} = 1 \;\text{ and }\; \tilde{d}_{zz} = 2.$$

Hence the Second Derivatives Test tells us that the function has a local minimum at $x = 0, z = 0$:

$$D = \tilde{d}_{xx} \cdot \tilde{d}_{zz} - \tilde{d}_{xz}^2 = 4 - 1 = 3 > 0 \;\text{ and }\; \tilde{d}_{xx} = 2 > 0$$

As this is the only minimum, it is the global minimum. As $y^2 = 9$ or $y = \pm 3$ we get the absolute minima at

$$P = (0, -3, 0) \;\text{ and }\; Q = (0, 3, 0). \text{ The distance is } d = 3.$$
