Math 8: Calculus in one and several variables
Spring 2017 - Homework 9

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keywords: *absolute maxima and minima, Lagrange multipliers*

exercise 1. The base B of an aquarium (without lid) is made of slate and the four sides are made out of glass. The volume of the aquarium is V. If slate costs five times as much as glass (per unit area), find the dimensions of the aquarium that minimize the cost of the material.

Solution: We denote by x the length, by y the width and by z the height of the aquarium. Then the volume V and area A of the aquarium are

$$V(x,y,z) = xyz = V \quad \text{and} \quad A(x,y,z) = xy + 2xz + 2yz.$$

The cost C of the aquarium is then as the cost of a unit area of the base is five times the cost of the sides:

$$C(x,y,z) = 5xy + 2xz + 2yz.$$

a) Find the minimum using the methods from Chapter 14.7 of the book.
Solution: Using the fact that $V = xyz$, hence $xy = \frac{V}{z}$ and $yz = \frac{V}{x}$, we can eliminate y from the cost:

$$C(x,z) := \frac{5V}{z} + 2xz + \frac{2V}{x}.$$

Now we can analyze the function C to find its minima. To this end we have to find the critical points, i.e. where the gradient is equal to $\langle 0,0\rangle$. We get

$$\mathbf{grad}(C) = \langle C_x, C_z\rangle = \langle 2z - \frac{2V}{x^2}, 2x - \frac{5V}{z^2}\rangle = \langle 0,0\rangle \Leftrightarrow 1.)\ z = \frac{V}{x^2} \quad \text{and} \quad 2.)\ 2x = \frac{5V}{z^2}.$$

Plugging $z = \frac{V}{x^2}$ into 2.) we get: $2x = \frac{5V\cdot x^4}{V^2}$. Hence $x = 0$ or $x = \left(\frac{2}{5}V\right)^{1/3}$. Clearly $x = 0$ means that $V = 0$, so this is not a valid solution. Then

$$x = \left(\frac{2}{5}V\right)^{1/3} \quad \text{in 1.) implies } z = \left(\frac{25}{4}V\right)^{1/3}. \quad \text{We set } P = \left(\left(\frac{2}{5}V\right)^{1/3}, \left(\frac{25}{4}V\right)^{1/3}\right)$$

For the second order derivatives we get:

$$C_{xx} = \frac{4V}{x^3}\ ,\ C_{xz} = C_{zx} = 2 \ \text{ and } \ C_{zz} = \frac{10V}{z^3}.$$

Hence the Second Derivatives Test tells us that the function has a local minimum at P:

$$D(P) = C_{xx}(P)\cdot C_{zz}(P) - C_{xz}^2(P) = 10\cdot\frac{8}{5} - 4 = 12 > 0 \ \text{ and } \ C_{xx}(P) = 10 > 0$$

As this is the only minimum, it is the global minimum. As $y = \frac{V}{xz}$ or $y = \left(\frac{2}{5}V\right)^{1/3}$ we get the absolute minima at Q with cost $C(Q)$ where

$$Q = \left(\left(\frac{2}{5}V\right)^{1/3}, \left(\frac{2}{5}V\right)^{1/3}, \left(\frac{25}{4}V\right)^{1/3}\right) \quad \text{and} \quad C(Q) \simeq 8.14 \cdot V^{2/3}.$$

b) Verify your result from part a) using the method of Lagrange multipliers.
Solution: In this case we have to minimize the cost $C(x, y, z)$ under the constraint $V(x, y, z) = xyz = V$.
Using the method of Lagrange multipliers we have to find x, y, z and λ, such that

$$I.)\ \mathbf{grad}(C)(x, y, z) = \lambda \cdot \mathbf{grad}(V)(x, y, z) \quad II.)\ xyz = V.$$

We get

$$\begin{aligned} \mathbf{grad}(C)(x, y, z) &= \langle C_x, C_y, C_z \rangle = \langle 5y + 2z, 5x + 2z, 2x + 2y \rangle, \\ \mathbf{grad}(V)(x, y, z) &= \langle V_x, V_y, V_z \rangle = \langle yz, xz, xy \rangle. \end{aligned}$$

This leads to the equations

$$\begin{aligned} 1.) \quad & 5y + 2z = \lambda yz \\ 2.) \quad & 5x + 2z = \lambda xz \\ 3.) \quad & 2x + 2y = \lambda xy \\ 4.) \quad & xyz = V \end{aligned}$$

Noting that $y, z, x \neq 0$ (since otherwise we have no aquarium!), we may multiply equations (1)-(3) by x, y, z, respectively, so that the right hand sides match:

$$\begin{aligned} 1.) \quad & 5xy + 2xz = \lambda xyz \\ 2.) \quad & 5xy + 2yz = \lambda xyz \\ 3.) \quad & 2xz + 2yz = \lambda xyz \end{aligned}$$

Comparing the left hand sides of (1) and (2), we see that $x = y$ and comparing the left hand sides of (2) and (3), we see that $5y = 2z$, or $z = \frac{5}{2}y$. Thus equation (4) says that $\frac{5}{2}y^3 = V$, i.e., $y = (\frac{2}{5}V)^{1/3}$. With $y = x$ and $z = \frac{5}{2}y$ we therefore obtain the same result as in part a).

Math 8: Calculus in one and several variables
Spring 2017 - Homework 9

Return date: not collected

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exercise 2. Find the dimensions of a box with volume $V = 1000\ cm^3$ that has minimal surface area.

Solution: We denote by x the length, by y the width and by z the height of the box. Then the volume V and area A of the box are

$$V(x,y,z) = xyz = 1000 \quad \text{and} \quad A(x,y,z) = 2xy + 2xz + 2yz.$$

a) Find or guess a solution using your geometric intuition.
   Solution: Intuitively the box with fixed volume that has maximal surface area is a cube. So we would expect that $x = y = z$, hence from the volume we get

$$V = x^3 = 1000 \quad \text{or} \quad x = y = z = 10\ cm\,.$$

b) Verify your result from part a) using the method of Lagrange multipliers.
   Solution: This is very similar to **exercise 1b)**. We have to minimize

$$A(x,y,z) \quad \text{under the constraint} \quad V(x,y,z) = xyz = 1000.$$

   Using the method of Lagrange multipliers we have to find x, y, z and λ, such that

$$I.)\ \mathbf{grad}(A)(x,y,z) = \lambda \cdot \mathbf{grad}(V)(x,y,z) \quad II.)\ xyz = 1000.$$

   We get

$$\begin{aligned} \mathbf{grad}(A)(x,y,z) &= \langle A_x, A_y, A_z\rangle = \langle 2y+2z, 2x+2z, 2x+2y\rangle, \\ \mathbf{grad}(V)(x,y,z) &= \langle V_x, V_y, V_z\rangle = \langle yz, xz, xy\rangle. \end{aligned}$$

   This leads to the equations

$$\begin{aligned} &1.) \quad 2y + 2z = \lambda yz \\ &2.) \quad 2x + 2z = \lambda xz \\ &3.) \quad 2x + 2y = \lambda xy \\ &4.) \quad xyz = 1000 \end{aligned}$$

   Solving 1.) and 2.) for λ we get, as $x, y, z \neq 0$:

$$1.)\ \frac{2y+2z}{yz} = \lambda = \frac{2x+2z}{xz}\ 2.) \quad \text{or} \quad \frac{2}{z} + \frac{2}{y} = \frac{2}{z} + \frac{2}{x} \quad \text{or} \quad x = y.$$

   Substituting $x = y$ in 3.) and 4.) we get

$$3.)\ 2x + 2y = \lambda xy \quad \text{or} \quad 4x = \lambda x^2 \quad \text{or} \quad \lambda = \frac{4}{x} \qquad 4.)\ xyz = V \quad \text{or} \quad z = \frac{V}{x^2}$$

Math 8: Calculus in one and several variables
Spring 2017 - Homework 9

Return date: not collected

Substituting $\lambda = \frac{4}{x}$ and $z = \frac{1000}{x^2}$ back in 2.) we get

$$2x + 2\frac{1000}{x^2} = \frac{4}{x} \cdot x \cdot \frac{1000}{x^2} \quad \text{or} \quad x^3 = 1000 \Leftrightarrow x = 10.$$

With $y = x$ and $z = \frac{1000}{x^2}$ we therefore obtain the same result as in part a):

$$x = y = z = 10\ cm\,.$$

exercise 3. Use the method of Lagrange multipliers to find the maximum and minimum of the function f subject to the given constraints

a) $f(x, y) = 3x + y$, if $x^2 + y^2 = 10$.
Solution: We have to minimize

$$f(x, y) = 3x + y \ \text{ under the constraint } \ g(x, y) = x^2 + y^2 = 10.$$

Using the method of Lagrange multipliers we have to find x, y and λ, such that

$$I.)\ \mathbf{grad}(f)(x, y) = \lambda \cdot \mathbf{grad}(g)(x, y) \ \ II.)\ x^2 + y^2 = 10.$$

We get

$$\begin{aligned} \mathbf{grad}(f)(x, y) &= \langle f_x, f_y \rangle = \langle 3, 1 \rangle, \\ \mathbf{grad}(g)(x, y) &= \langle g_x, g_y \rangle = \langle 2x, 2y \rangle. \end{aligned}$$

This leads to the equations

$$\begin{aligned} 1.) &\quad 3 = \lambda 2x \\ 2.) &\quad 1 = \lambda 2y \\ 3.) &\quad x^2 + y^2 = 10 \end{aligned}$$

Now $x, y \neq 0$ as this would contradict 1.) and 2.). Solving 1.) and 2.) for λ we get:

$$1.)\ \frac{3}{2x} = \lambda = \frac{1}{2y}\ 2.) \ \text{ or } \ 6y = 2x \ \text{ or } \ x = 3y.$$

Substituting $x = 3y$ in 3.) we get

$$3.)\ x^2 + y^2 = 10 \ \text{ or } \ 9y^2 + y^2 = 10 \ \text{ or } \ y = \pm 1.$$

As $x = 3y$ we get the two possible extrema: $P = (-3, -1)$,$(\lambda = -\frac{1}{2})$ and $Q = (3, 1)$, $(\lambda = \frac{1}{2})$. Evaluating $f(P)$ and $f(Q)$ we get

$$\text{Minimum of } f : -10 = f(-3, -1) \ \text{ at } \ P \qquad \text{Maximum of } f : 10 = f(3, 1) \ \text{ at } \ Q.$$

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b) $f(x,y,z) = xy^2z$, if $x^2+y^2+z^2=4$.
Solution: We have to minimize

$$f(x,y,z) = xy^2z \text{ under the constraint } g(x,y,z) = x^2+y^2+z^2 = 4.$$

For the gradients of these functions we get

$$\begin{aligned}\mathbf{grad}(f)(x,y,z) &= \langle f_x, f_y, f_z\rangle = \langle y^2z, 2xyz, xy^2\rangle,\\ \mathbf{grad}(g)(x,y,z) &= \langle g_x, g_y, g_z\rangle = \langle 2x, 2y, 2z\rangle.\end{aligned}$$

Using the method of Lagrange multipliers this leads to the equations

$$\begin{aligned}1.)&\quad y^2z = \lambda 2x\\ 2.)&\quad 2xyz = \lambda 2y\\ 3.)&\quad xy^2 = \lambda 2z\\ 4.)&\quad x^2+y^2+z^2 = 4\end{aligned}$$

Note that if any of x, y or z is zero, then $f(x,y,z)=0$. Since f has both positive and negative values on the sphere $x^2+y^2+z^2=4$, no such point can be a maximum or minimum. Thus we may assume that x, y and z are all non-zero. We multiply equations 1), 2) and 3) by $2x$, y and $2z$, respectively, so that all the left-hand sides are the same:

$$\begin{aligned}1.)&\quad 2xy^2z = \lambda 4x^2\\ 2.)&\quad 2xy^2z = \lambda 2y^2\\ 3.)&\quad 2xy^2z = \lambda 4z^2\end{aligned}$$

Comparing the right-hand sides of the three equations, we see that $y^2 = 2x^2$ and $z^2 = x^2$. Plugging into equation (4), we thus get $x^2+2x^2+x^2=4$, i.e., $x^2=1$ and $x=\pm 1$. Then $y^2=2$ and $z^2=1$, so $y=\pm\sqrt{2}$ and $z=\pm 1$. We thus get eight constrained critical points $(1,\sqrt{2},1)$, $(1,\sqrt{2},-1)$, $(1,-\sqrt{2},1)$, $(1,-\sqrt{2},-1)$, $(-1,\sqrt{2},1)$, $(-1,\sqrt{2},-1)$, $(-1,-\sqrt{2},1)$, and $(-1,-\sqrt{2},-1)$. For each of these points, $xy^2z=\pm 2$. So the maximum of f subject to the constraint is 2 and the minimum is -2.

exercise 4. Find the absolute maxima and minima of the function f in the region D.

a) $f(x,y) = 2x^2+3y^2-4x-5$, where $D: x^2+y^2\le 16$
Solution: We have to find all possible extrema of the function in the region D. The possible

extrema in the interior of D are the critical points. These satisfy $\mathbf{grad}(f)(a,b) = \langle 0,0\rangle$. The other possible extrema are the extreme values of f on the boundary ∂D of D.
We first check for critical points in the interior:

$$\mathbf{grad}(f) = \langle f_x, f_y\rangle = \langle 4x-4, 6y\rangle = \langle 0,0\rangle \Leftrightarrow x=1, y=0 \text{ and } f(1,0) = -7.$$

Hence the only critical point is $P = (1,0)$ with $f(1,0) = -7$.

We now look at the boundary $\partial D : x^2 + y^2 = 16$. There are several ways of finding extreme values on the boundary. We can use the method from Chapter 14.7 or Lagrange multipliers. As it is easier in this case, we will use the first.
The coordinates of a point $Q = (x,y)$ on the boundary ∂D satisfy
$x^2 + y^2 = 16 \Leftrightarrow y^2 = 16 - x^2$. We can substitute this into our function $f(x,y)$.

$$\partial D: \; f(x,y) = f(x) = 2x^2 + 3(16-x^2) - 4x - 5 = -x^2 - 4x + 43, \quad \text{where } x \in [-4,4].$$

The interval for x comes from the condition that $x^2 + y^2 = 16$ or $x^2 = 16 - y^2 \leq 16$.
The function $f(x)$ might have an extremum in the interior of the interval $[-4,4]$.
We get with $f(x) = -x^2 - 4x + 43$:

$$\frac{\partial}{\partial x} f(x) = 0 \Leftrightarrow -2x - 4 = 0 \text{ hence } x = -2. \text{ Then } y^2 = 16 - x^2 \Leftrightarrow y = \pm\sqrt{12}.$$
$$\text{So } Q = (-2, -\sqrt{12}) \text{ and } R = (-2, \sqrt{12}) \text{ and } f(Q) = f(R) = 47.$$

Another extreme value might be on the endpoints of the interval $[-4,4]$. We get:

$$x = -4 : S = (-4,0),\; f(-4,0) = 43 \text{ and } x = 4 : T = (4,0),\; f(4,0) = 11.$$

Comparing the values for the possible extrema, we get:

Minimum of $f : -7 = f(1,0)$ at P. Maximum of $f : 47 = f(-2, \pm\sqrt{12})$ at Q and R .

b) $f(x,y,z) = x^2 + 2y^2 + 3z^2$, where $D : x + y + 3z = 10$.
Solution: This time we use the method of Lagrange multipliers. We have to minimize

$$f(x,y,z) = x^2 + 2y^2 + 3z^2 \text{ under the constraint } g(x,y,z) = x + y + 3z = 10.$$

For the gradients of these functions we get

$$\begin{aligned} \mathbf{grad}(f)(x,y,z) &= \langle f_x, f_y, f_z\rangle = \langle 2x, 4y, 6z\rangle, \\ \mathbf{grad}(g)(x,y,z) &= \langle g_x, g_y, g_z\rangle = \langle 1,1,3\rangle. \end{aligned}$$

Math 8: Calculus in one and several variables
Spring 2017 - Homework 9

Return date: not collected

Using the method of Lagrange multipliers this leads to the equations

$$\begin{aligned} &1.) \quad 2x = \lambda \\ &2.) \quad 4y = \lambda \\ &3.) \quad 6z = 3\lambda \\ &4.) \quad x + y + 3z = 10 \end{aligned}$$

Comparing 1.),2.) and 3.) with respect to λ we get $x = z = 2y$. Substituting x and z in 4.) we get:

$$4.)\ x + y + 3z = 10 \ \text{ hence } \ 2y + y + 6y = 10 \ \text{ or } \ y = \frac{10}{9}.$$

Hence the only extreme value occurs if $x = z = \frac{20}{9}, y = \frac{10}{9}, \lambda = \frac{40}{9}$.

$$\text{We set } \ P = (\frac{20}{9}, \frac{10}{9}, \frac{20}{9}) \ \text{ and get } \ f(P) = \frac{200}{9} \simeq 22.22.$$

Taking another point $Q = (0, 10, 0)$ in the plane defined by $x + y + 3z = 10$, we see that $f(Q) = 200$. Hence there must be a minimum in P.