CHAIN RULE HANDOUT

MAY 10, 2019

Theorem (Chain Rule, case 1)**.** Suppose that $x = x(t)$ and $y = y(t)$ are differentiable functions of t and $z = z(x, y)$ is a differentiable function of x and y. Then $z(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

where $\frac{dz}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$ are evaluated at t and $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are evaluated at (x, y).

Theorem (Chain Rule, case 2)**.** Suppose that $x = x(u, v)$ and $y = y(u, v)$ are differentiable functions of u and v, and $z = z(x, y)$ is a differentiable function of x and y. Then $z(x(u, v), y(u, v))$ is a differentiable function of u and v and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

where each partial derivative is evaluated at the appropriate point (u, v) or (x, y).

Theorem (Chain Rule, general case)**.** Let $w = w(x_1, \ldots, x_m)$ be a differentiable function of m variables, and for each $i = 1, \ldots, m$, let $x_i = x_i(t_1, \ldots, t_n)$ be a differentiable function of n variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for each $j = 1, \ldots, n$.

Exercise 1.

(a) Let $z = \sin(\theta)\cos(\varphi)$, $x = st^2$, $y = s^2t$. Compute $\dfrac{\partial z}{\partial s}$.

(b) Let $w = xe^{y/z}$, $x = t^2$, $y = 1 - t$ and $z = 1 + 2t$. Compute dw/dt.

(c) Let $u = x^2 + yz$, $x = st\cos(\theta)$, $y = st\sin(\theta)$, and $z = s + t$. Find $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$, and $\frac{\partial u}{\partial \theta}$ when $s = 2, t = 3$, and $\theta = 0$.

Exercise 2. Let C be the curve defined by $\sin(x) + \cos(y) = \sin(x)\cos(y)$. Use partial derivatives to find dy/dx.