Handout: Properties of power series

Theorem (Properties of power series)**.** Suppose that

$$\sum_{n=0}^{\infty} c_n x^n \text{ converges to } f(x) \text{ for all } x \text{ in an interval } I_1$$

and

$$\sum_{n=0}^{\infty} d_n x^n \text{ converges to } g(x) \text{ for all } x \text{ in an interval } I_2.$$

Then:

(a) $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f(x) \pm g(x)$ for x values in the overlap of I_1 and I_2.

(b) $\sum_{n=0}^{\infty} b x^m c_n x^n$ converges to $b x^m f(x)$ for x values in I_1 (where b is a fixed value and $m \geq 0$).

(c) $\sum_{n=0}^{\infty} c_n (b x^m)^n$ converges to $f(b x^m)$ as long as $b x^m$ is in I_1 (where b is a fixed value and $m \geq 0$).

(d) $\left(\sum_{n=0}^{\infty} c_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} d_n x^n\right)$ converges to $f(x) \cdot g(x)$ for x in the overlap of I_1 and I_2. (**Warning:** we cannot multiply series by just multiplying the corresponding terms! We have to distribute.)

(e) $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ for x values in I_1. (Take the derivative of each term.)

(f) $\int f(x)\, dx = C + \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}$ for x values in I_1. (Take the antiderivative of each term.)

Parts (e) and (f) are called "term-by-term" differentiation and integration.

Note: Parts (a), (d), (e), and (f) still hold if the center of the power series is nonzero, just replace x with $(x - a)$. For parts (a) and (d), both series must have the same center.

Exercises:

(1) Write $\dfrac{2x^2}{1-x}$ as a power series. What is its interval of convergence?

(2) Write $\dfrac{7}{1+3x}$ as a power series. What is its interval of convergence?

(3) Add your results from parts (1) and (2). What is the resulting function? What is the interval of convergence?

(4) Find the first 3 terms (constant, x, x^2) of the series obtained by multiplying

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n}} \quad \text{and} \quad \sum_{n=0}^{\infty} n2^n x^n.$$

Hint: Write out the first few terms of each series and FOIL/distribute. You need at least 3 terms of each. (Why?)

(5) What is the interval of convergence of the resulting series in part (4)?
Hint: Find the intervals of convergence of each series and use part (d) of the theorem.

(6) Find a series representing the function $6\ln(1-x)$.
Hint: $\frac{d}{dx}(6\ln(1-x)) = \frac{-6}{1-x}$. Apply part (f) of the theorem.

(7) Find a series representing the function $\dfrac{x}{(1+2x^2)^2}$.
Hint: Take the antiderivative of the function, then apply part (e) of the theorem.