Handout: Sequences and Series

**Theorem** (Algebraic Limit Laws)**.** Consider two **convergent** sequences $\{a_n\}$ and $\{b_n\}$ where $\{a_n\}$ converges to $A$ and $\{b_n\}$ converges to $B$. Let $c$ be any real number. The following properties hold:

$$\lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} c\, a_n = cA$$

$$\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$$

$$\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$$

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}, \text{ as long as } b_n \neq 0 \text{ and } B \neq 0$$

**Theorem.** Consider a **convergent** sequence $\{a_n\}$ which converges to $A$. Let $f(x)$ be a function which is continuous at $A$. Then the sequence $\{f(a_n)\}$ also converges and, in fact, converges to $f(A)$.

**Big Idea:** If we can break a sequence down into pieces that we know converge, then we can determine the limit of the sequence.

**Exercise 1.** Consider the sequence

$$\left\{1, \frac{4}{3}, 1, \frac{16}{27}, \frac{25}{81}, \ldots\right\}.$$

(a) Find the next two terms in the sequence.
(b) Find the general term $a_n$ for the sequence. (Choose your own starting index.)

**Exercise 2.** Consider the sequence defined by the following recurrence relation:

$$a_1 = 4,\ a_{n+1} = \frac{a_n}{a_n - 1}.$$

(a) Find the first five terms in the sequence.
(b) Find an explicit formula for the general term $a_n$.
(Hint: Consider defining $a_{2n}$ and $a_{2n+1}$ separately.)

**Exercise 3.** Determine whether each of the following sequences converges or diverges. Use facts that you know from limits of functions along with the algebraic limit laws.

(a) $a_n = \dfrac{3 + 5n^2}{n^2 + 2n}$

(b) $\{5\sin(n)\}$

(c) $b_n = \ln\left(\dfrac{n+1}{n}\right) + 6 - \cos(2\pi n)$

**Theorem** (Algebraic Properties of Convergent Series)**.** Consider two **convergent** series $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Let $c$ be any real number. The following properties hold:

$$\sum_{n=1}^{\infty} c\, a_n = cA$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$$

**Note:** It is not important here that the series start at $n = 1$, but it *is* important that they have the *same* starting index. So, if you would like to add two series with different starting indices, you should shift the index of one to match the other first.

**Exercise 4.** Consider the series

$$\sum_{n=1}^{\infty} \ln(n+1) - \ln(n).$$

(a) Compute the first 5 terms of the sequence of partial sums $\{S_N\}_{N=1}^{\infty}$.
(b) Find the general term $S_N$ of the sequence of partial sums.
(c) Does the sequence converge or diverge? If it converges, what does it converge to?
(d) Does the series converge? If it converges, what does it converge to?