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Multiplication of vectors

If $\vec{v} =< v_1, v_2, v_3 >$ and $\vec{w} =< w_1, w_2, w_3 >$ are two vectors then the *dot product* of $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Note: The dot product is sometimes also called the *scalar product* or *inner product*.

Properties of the dot product

1. The dot product calculates lengths:

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

2. The dot product calculates the angle between vectors:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta)$$

   where θ is the angle between $\vec{v}$ and $\vec{w}$. We can also rewrite this formula as:

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$$

3. We say two vectors are *orthogonal* (or *perpendicular*) if $\theta = \frac{\pi}{2}$ and that the two vectors are *parallel* if $\theta = 0$.

4. From this, we see that two vectors are *orthogonal* if

$$\vec{v} \cdot \vec{w} = 0$$

Examples

Find the cosine of the angle between the following vectors:

- $< 3, 4 >$ and $< 5, 12 >$
- $< 1, 2, 3 >$ and $< 4, 0, -1 >$
- Find a vector that is perpendicular to both $< -1, 1, 0 >$ and $< 4, 0, -1 >$.

Proof of the angle formula

This follows from the law of cosines: If a triangle is determined by points O, A and B and θ is the angle at the vertex O then

$$|\overrightarrow{AB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}|\cos(\theta)$$

If $\vec{v} = \overrightarrow{OA}$ and $\vec{w} = \overrightarrow{OB}$ then $\vec{v} - \vec{w}$ is a copy of $\overrightarrow{AB}$. So, the formula becomes

$$\begin{aligned}
|\vec{v} - \vec{w}|^2 &= |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos(\theta) \\
-|\vec{v} - \vec{w}|^2 + |\vec{v}|^2 + |\vec{w}|^2 &= 2|\vec{v}||\vec{w}|\cos(\theta) \\
|\vec{v}|^2 + |\vec{w}|^2 - (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) &= 2|\vec{v}||\vec{w}|\cos(\theta) \\
|\vec{v}|^2 + |\vec{w}|^2 - |\vec{v}|^2 - |\vec{w}|^2 + 2\vec{v} \cdot \vec{w} &= 2|\vec{v}||\vec{w}|\cos(\theta) \\
\vec{v} \cdot \vec{w} &= |\vec{v}||\vec{w}|\cos(\theta)
\end{aligned}$$

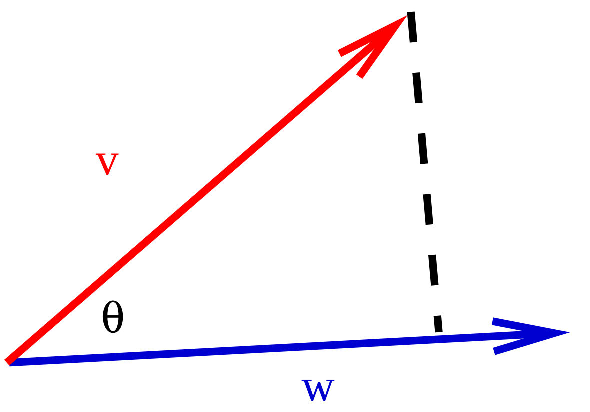
Projections

It is often useful to be able to project one vector onto another. We have two formulas that help us calculate such a projection:

$$proj_{\vec{v}}\vec{w} = \left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}|^2}\right)\vec{v}$$

The scalar projection, or component, of $\vec{w}$ onto $\vec{v}$ is

$$comp_{\vec{v}}\vec{w} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|}$$



Examples

- Find the scalar and vector projections of $\vec{w} =< 1, 1, 2 >$ onto $\vec{v} =< -2, 3, 1 >$.
- Let T be a triangle with vertices at $P = (1, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 1, 1)$. What is the cosine of the angle between the side $\overrightarrow{PQ}$ and $\overrightarrow{PR}$?

Cross product

There is another way to multiply vectors: given two vectors $\vec{v} =< v_1, v_2, v_3 >$ and $\vec{w} =< w_1, w_2, w_3 >$, their *cross product* is

$$\vec{v} \times \vec{w} =< v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 >$$

Note: we can also phrase this formula in terms of determinants:

$$\vec{v} \times \vec{w} = det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

Properties

- The vector $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.
- Which way? Use the right hand rule.
- $$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\theta)$$
- Two nonzero vectors are parallel if $|\vec{v} \times \vec{w}| = 0$
- $|\vec{v} \times \vec{w}|$ is equal to the area of a the parallelogram determined by $\vec{v}$ and $\vec{w}$.
- The volume of the parallelopiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is
$$|\vec{u} \cdot (\vec{v} \times \vec{w})|$$