Math 8, Winter 2005

Scott Pauls

Dartmouth College, Department of Mathematics

1/12/05

With Acroread, CTRL-L switch between full screen and window mode

Elliptical Integrals

Another type of common integral that comes up in applications are the elliptical integrals. These types of integrals include, for example, the integral arising when trying to compute the area of a circle of radius r:

$$\int \sqrt{r^2 - x^2}\, dx$$

Elliptical Integrals

Another type of common integral that comes up in applications are the elliptical integrals. These types of integrals include, for example, the integral arising when trying to compute the area of a circle of radius r:

$$\int \sqrt{r^2 - x^2}\, dx$$

- How do we compute these types of integrals? Substitution, integration by parts, etc. do not work...

Elliptical Integrals

Another type of common integral that comes up in applications are the elliptical integrals. These types of integrals include, for example, the integral arising when trying to compute the area of a circle of radius r:

$$\int \sqrt{r^2 - x^2}\, dx$$

- How do we compute these types of integrals? Substitution, integration by parts, etc. do not work...
- As before, use trigonometric identities, e.g.

$$\sin^2(x) + \cos^2(x) = 1$$

Area of a circle

- Area of the top half of a circle of radius r is

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

Area of a circle

- Area of the top half of a circle of radius r is

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

- Problem: the square root makes finding an anti-derivative difficult.

Area of a circle

- Area of the top half of a circle of radius r is

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

- Problem: the square root makes finding an anti-derivative difficult.

- Trick: convert the term inside the square root into a square. Let

$$x = r\cos(\theta)$$

  Then,

$$r^2 - x^2 = r^2 - r^2\cos^2(\theta) = r^2\sin^2(\theta)$$

Area of a circle

- Area of the top half of a circle of radius r is

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

- Problem: the square root makes finding an anti-derivative difficult.

- Trick: convert the term inside the square root into a square. Let

$$x = r\cos(\theta)$$

Then,

$$r^2 - x^2 = r^2 - r^2\cos^2(\theta) = r^2\sin^2(\theta)$$

- That implifies the integrand, but we need to rewrite dx in terms of θ:

$$dx = d(r\cos(\theta)) = -r\sin(\theta)d\theta$$

Area of a circle

- Via this *trigonometric substitution*, we have that

$$\int \sqrt{r^2 - x^2}\, dx = -\int \sqrt{r^2 \sin^2(\theta)} r \sin(\theta) d\theta$$

Note: $r^2 \sin^2(x) \geq 0$.

Area of a circle

- Via this *trigonometric substitution*, we have that

$$\int \sqrt{r^2 - x^2}\; dx = -\int \sqrt{r^2 \sin^2(\theta)}\, r \sin(\theta) d\theta$$

Note: $r^2 \sin^2(x) \geq 0$.

- Evaluate this integral:

$$\begin{aligned} \int -\sqrt{r^2 \sin^2(\theta)}\, r \sin(\theta)\; d\theta &= -r^2 \int \sin^2(\theta)\; d\theta \\ &= -r^2 \left(\frac{\theta}{2} - \frac{\sin(2\theta)}{4} \right) + C \end{aligned}$$

Area of a circle

- Via this *trigonometric substitution*, we have that

$$\int \sqrt{r^2 - x^2}\, dx = -\int \sqrt{r^2 \sin^2(\theta)}r\sin(\theta)d\theta$$

Note: $r^2 \sin^2(x) \geq 0$.

- Evaluate this integral:

$$\int -\sqrt{r^2 \sin^2(\theta)}r\sin(\theta)\, d\theta = -r^2 \int \sin^2(\theta)\, d\theta$$

$$= -r^2 \left(\frac{\theta}{2} - \frac{\sin(2\theta)}{4} \right) + C$$

- Rewrite answer in terms of x:

$$x = r\cos(\theta) \implies \theta = \arccos\left(\frac{x}{r}\right)$$

Area of a circle

- So, back-substituting and evaluating between $x = -r$ and $x = r$, we have

$$-r^2\left(\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\right)\Bigg|_{-r}^{r} = -r^2\,\frac{\arccos\left(\frac{x}{r}\right)}{2} - r^2\sin\left(2\arccos\left(\frac{x}{r}\right)\right)\Bigg|_{-r}^{r}$$

$$= -\left(r^2\,\frac{\arccos\left(\frac{r}{r}\right)}{2} - r^2\,\frac{\arccos\left(\frac{-r}{r}\right)}{2}\right)$$

$$-\left(r^2\sin\left(2\arccos\left(\frac{r}{r}\right)\right) - r^2\sin\left(2\arccos\left(\frac{-r}{r}\right)\right.\right.$$

$$= -r^2(\arccos(1) - \arccos(-1)$$

$$-\left(\sin(2\arccos(1))\right) - \sin(2\arccos(-1)))$$

$$= r^2\,\frac{\pi}{2}$$

- $$\int x^3\sqrt{9-x^2}\,dx$$
- Substitute: $x = 3\sec(x)$,
$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$$
- Substitute: $x = 2\tan(x)$,
$$\int \frac{x^3}{\sqrt{x^2+4}}$$

Rational functions

If $p(x)$ and $q(x)$ are polynomials, how do we integrate

$$\int \frac{p(x)}{q(x)}\,dx$$

Cases we know:

$$\int \frac{1}{ax+b}\,dx$$

$$\int \frac{x}{ax^2+b}\,dx$$

We'll try to rewrite the integral in terms of these types of easier integrals.

Rational functions

If $p(x)$ and $q(x)$ are polynomials, how do we integrate

$$\int \frac{p(x)}{q(x)}\,dx$$

Cases we know:

$$\int \frac{1}{ax+b}\,dx$$

$$\int \frac{x}{ax^2+b}\,dx$$

We'll try to rewrite the integral in terms of these types of easier integrals.

How? If we have two fractions, e.g.

$$\frac{1}{x+1}+\frac{1}{x-1}$$

we often simplify to get a common denominator:

$$\frac{(x-1)+(x+1)}{(x+1)(x-1)}=\frac{1}{x^2-1}$$

Partial Fractions

Partial Fractions is a technique where we undo the process of finding common denomenators.

- $$\int \frac{1}{\sqrt{1-x^2}}\,dx$$
- $$\int \frac{x-9}{x^2+3x-10}\,dx$$
- $$\int \frac{x^2+2x-1}{x^3-x}\,dx$$