Math 8, Winter 2005

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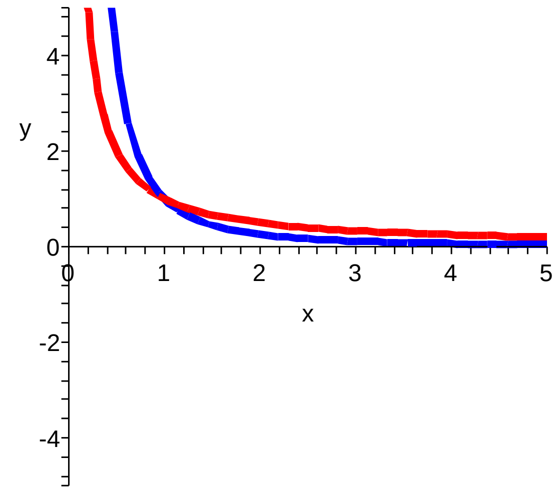
Introduction

Introduction

- Last class, we looked at two basic improper integrals:

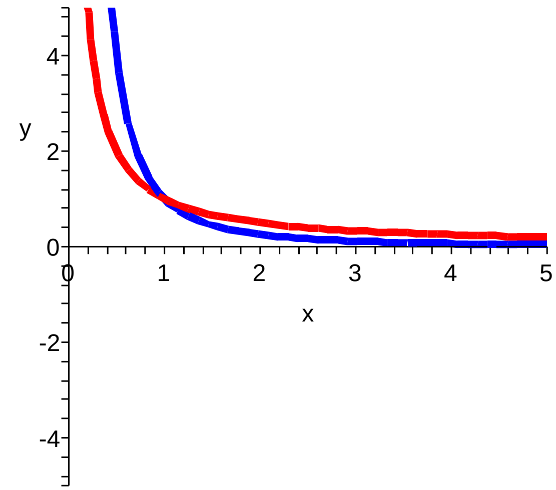
$$\int_1^\infty \frac{1}{x}\,dx \quad \text{diverges}$$

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$



- Key point: $\frac{1}{x^2}$ tends to zero much faster than $\frac{1}{x}$.

Introduction



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- Key point: $\frac{1}{x^2}$ tends to zero much faster than $\frac{1}{x}$.

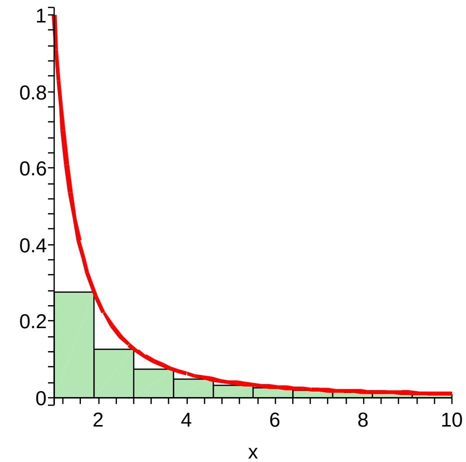
- Look at the integral as measuring area under the curve from $x = 1$ to $x = n + 1$ with n boxes.

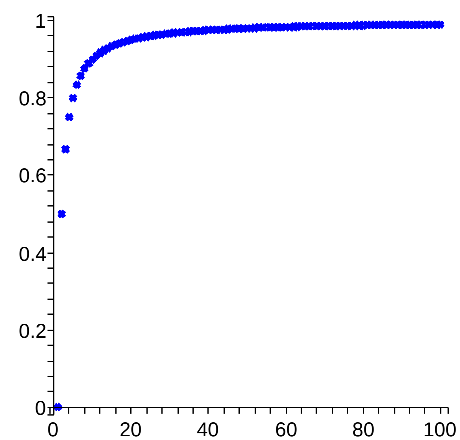
Counting area: sequences

- Consider $\int_1^\infty \frac{1}{x^2}\,dx$ first.
- $$a_n = \int_1^{n+1} \frac{1}{x^2}\,dx = 1 - \frac{1}{(n+1)}$$
- $\{a_n\}$ gives a *sequence* of numbers:

$$\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$$

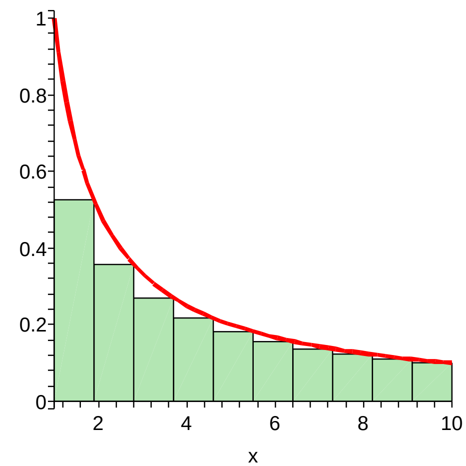
- Knowing the value of the integral is 1, we know that the numbers in theis sequence tend to 1. We say the sequence *converges* to 1.





Counting area: sequences

- $$a_n = \int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1) - \ln(1) = \ln(n+1)$$



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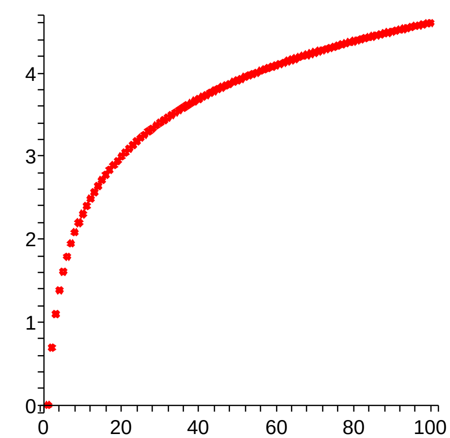


- Knowing that the area under the curves tends towards ∞, we know that the numbers in theis sequence tend to ∞ as well. We say the sequence *diverges*.

Convergent sequences

Definition: A sequence $\{a_n\}$ converges to a limit L if, given any $\varepsilon > 0$, there exits an integer N (depending on ε, so that

$$|a_n - L| < \varepsilon \quad \text{for all } n \geq N$$

Convergent sequences

Theorem: If

$$\lim_{x \to \infty} f(x) = L$$

and $f(n) = a_n$ when n is an integer then

$$\lim_{n \to \infty} a_n = L$$

i.e. a_n converges to L.

Limit Laws

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is constant then

- $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p \quad$ if $p > 0$ and $a_n > 0$

Squeeze Theorem

Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = L$ and $\{c_n\}$ is a sequence so that $a_n \leq c_n \leq b_n$ then $\lim_{n\to\infty} c_n = L$.

Examples

- $$\lim_{n\to\infty} \frac{\ln(n)}{n} = ?$$
- For which values of r does
$$\lim_{n\to\infty} r^n$$
exist?
- $$\lim_{n\to\infty} \frac{3+5n^2}{n+n^2} = ?$$

Series

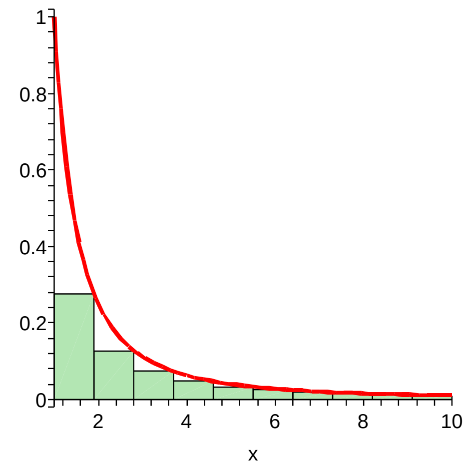
- Consider $\int_1^\infty \frac{1}{x^2}\,dx$ again.
- $\Delta x = \frac{n+1-1}{n} = 1$,

$$\int_1^{n+1} \frac{1}{x}\,dx \sim \sum_{i=1}^{n} \frac{1}{(i+1)^2}$$

- So,

$$\int_1^{\infty} \frac{1}{x^2}\,dx \geq \sum_{i=1}^{\infty} \frac{1}{(i+1)^2}$$

- Knowing the value of the integral is 1, we know that the numbers in theis sequence tend to 1. We say the sequence *converges* to 1.



Series

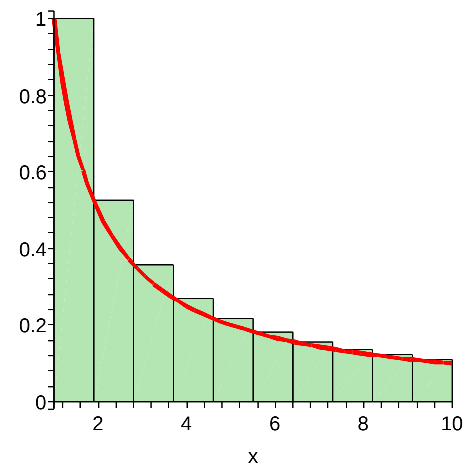
- Similarly,

$$\int_1^{\infty} \frac{1}{x}\, dx \leq \sum_{i=1}^{\infty} \frac{1}{i}$$

- Since the integral diverges, this shows that

$$\sum_{i=1}^{\infty} \frac{1}{i}$$

diverges as well.



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