Worksheet #10

Find the radius and interval of convergence of the series.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)\sqrt{n}}{5\sqrt{n+1}}\right|$$
$$= \frac{|2x-1|}{5}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|2x-1|}{5}$$

Thus the series will converge for $\frac{|2x-1|}{5} < 1$. Rewriting this, we find we need $|x - 1/2| < 5/2$. The radius of convergence is $R = 5/2$. To determine interval of convergence, we must check the endpoints of $-2 < x < 3$.

When $x = -2$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{1/2}}$ which converges by the alternating series test. When $x = 3$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$ which diverges by the p-test.

Thus the interval of convergence is $-2 \le x < 3$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^{1/3}}$

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^{1/3}}\frac{n^{1/3}}{x^n}\right| = \lim_{n\to\infty}\left|x\left(\frac{n}{(n+1)}\right)^{1/3}\right|$$
$$= |x|\lim_{n\to\infty}\left(\frac{n}{(n+1)}\right)^{1/3} = |x|$$

Thus the series will converge for $|x| < 1$. The radius of convergence is $R = 1$. To determine interval of convergence, we must check the endpoints.

When $x = 1$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{1/3}}$ which converges by the alternating series test. When $x = -1$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{n^{1/3}}$ which diverges by the p-test.

Thus the interval of convergence is $-1 < x \le 1$.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-3)^n}{2n+1}$

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2n+2}\frac{2n+1}{(x-3)^n}\right| = \lim_{n\to\infty}\left|(x-3)\frac{2n+1}{2n+2}\right|$$
$$= |(x-3)|\lim_{n\to\infty}\left(\frac{2n+1}{2n+2}\right) = |x-3|$$

Thus the series will converge for $|x-3| < 1$. The radius of convergence is $R = 1$. To determine interval of convergence, we must check the endpoints of $2 < x < 4$.

When $x = 4$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ which converges by the alternating series test. When $x = 2$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ which diverges by comparison test with $\frac{1}{n}$.

Thus the interval of convergence is $2 < x \leq 4$.

(4) $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

Solution: We use the ratio test.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| &= \lim_{n\to\infty}|(n+1)(2x-1)| \\ &= |2x-1| \lim_{n\to\infty}(n+1)\end{aligned}$$

This goes to infinity for all x not equal to $1/2$. Thus the radius of convergence is $R = 0$ and the interval of convergence is the point $x = 1/2$.