Worksheet #13

(1) Use the bionomial series to expand $f(x) = (8+x)^{1/3}$ as a power series. State the radius of convergence.
Solution: For this problem we can use the binomial series.

$$(8+x)^{1/3} = 2(1+\frac{x}{8})^{1/3} = 2\sum_{n=0}^{\infty}\binom{1/3}{n}\left(\frac{x}{8}\right)^n$$

Interval of convergence is $-8 < x < 8$.

(2) Use a known Mclaurin series to obtain the Mclaurin series for $f(x) = x\cos(\frac{x}{2})$.
Solution: We know the Mclaurin series for $\cos(y)$.

$$\cos(y) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}y^{2n}$$

Thus $\cos\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}2^{-2n}x^{2n}$.

Therefore $x\cos\left(\frac{x}{2}\right) = x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}2^{-2n}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}2^{-2n}x^{2n+1}$.

(3) Evaluate the indefinite integral as an infinite series.

$$\int \frac{e^x - 1}{x}dx$$

Solution: We know the Mclaurin series for e^x is $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.
Now

$$\frac{e^x-1}{x} = \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

Thus

$$\begin{aligned}
\int \frac{e^x-1}{x}dx &= \int \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x}dx \\
&= \int 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots dx \\
&= x + \frac{x^2}{(2)2!} + \frac{x^3}{(3)3!} + \frac{x^4}{(4)4!} + \cdots + C \\
&= \sum_{n=1}^{\infty}\frac{x^n}{(n)n!} + C
\end{aligned}$$

(4) Use series to evaluate

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2}$$

Solution: Note that

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x}dx = \int \frac{1}{1-(-x)}dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n x^n dx \\
&= \sum_{n=0}^{\infty} (-1)^n \int x^n dx \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}
\end{aligned}$$

Thus $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$. Therefore

$$\begin{aligned}
\frac{x - \ln(1+x)}{x^2} &= \frac{x - (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots)}{x^2} \\
&= \frac{\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \cdots)}{x^2} \\
&= \frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} + \cdots
\end{aligned}$$

Thus

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \frac{1}{2}.$$

(5) Find the sum of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$$

Solution: We know $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Thus we can rewrite the series in the following way.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1} \\
&= \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}
\end{aligned}$$