Worksheet #21

(1) Find the limit if it exist

- $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$
  Solution: If we let $y = x$ then,
  $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}.$$
  If we let $x = 0$ then,
  $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0.$$
  Since we get two different values, the limit does not exist.

- $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{3x^2+3y^2}$
  Solution: We use the Mclaurin series for $\sin(w)$.
  $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{3x^2+3y^2} = \frac{1}{3}\frac{x^2+y^2+(x^2+y^2)^3/3!+\cdots}{x^2+y^2} = \frac{1}{3}$$

- $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^4-y^4}$
  Solution: We can rewrite the function
  $$\frac{x^2+y^2}{x^4-y^4} = \frac{x^2+y^2}{(x^2-y^2)(x^2+y^2)} = \frac{1}{(x^2-y^2)}$$
  If we take the limit along $(x, 0)$, we get
  $$\lim_{x\to 0} \frac{1}{x^2} \to \infty$$
  Now taking the limit along $(0, y)$, we get
  $$\lim_{y\to 0} \frac{1}{-y^2} \to -\infty$$
  The two limits do not match, thus the limit does not exist.

(2) Determine the largest set where the function is continuous.

- $f(x, y) = (4 - x^2 - y^2)^{-1/2}$
  Solution: $\{(x, y) : 4 > x^2 + y^2\}$ This corresponds to the interior of a circle centered at the orgin with radius 2.

- $f(x, y) = \ln(1 - x^2 - y^2)$
  Solution: $\{(x, y) : 1 > x^2 + y^2\}$ This corresponds to the interior of a circle centered at the orgin with radius 1.

- $f(x, y) = \frac{x^3+xy-5}{x^2+y^2+1}$
  Solution: This is continuous for all (x, y) in $\mathbf{R}^2$.