Worksheet #3

Perform the indicated integrations.

(1) $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}dx$

Solution:

$$\int \frac{\sqrt{4-x^2}}{x^2}dx = 2\int \frac{\sqrt{1-(\frac{x}{2})^2}}{x^2}dx$$

Let $\frac{x}{2} = \sin\theta$, thus $x = 2\sin\theta$ and $dx = 2\cos\theta d\theta$.

$$\begin{aligned}
\int \frac{\sqrt{4-x^2}}{x^2}dx &= 2\int \frac{\sqrt{1-(\sin\theta)^2}}{4\sin^2\theta}2\cos\theta d\theta \\
&= \int \frac{\cos\theta}{\sin^2\theta}\cos\theta d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}d\theta \\
&= \int \cos^2\theta(\sin^2\theta)^{-1}d\theta \\
&= \int (1-\sin^2\theta)(\sin^2\theta)^{-1}d\theta \quad \text{write/in/terms/of/sine} \\
&= \int \left((\sin^2\theta)^{-1}-1\right)d\theta \\
&= \int \left(\csc^2\theta-1\right)d\theta \\
&= -\cot\theta-\theta+C = -\frac{\sqrt{4-x^2}}{x} - \arcsin(\frac{x}{2})+C
\end{aligned}$$

(2) $\displaystyle\int \frac{dx}{x\sqrt{x^2-9}}$

Solution:

$$\int \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}\int \frac{dx}{x\sqrt{(\frac{x}{3})^2-1}}$$

Let $\frac{x}{3} = \sec\theta$, thus $x = 3\sec\theta$ and $dx = 3\sec\theta\,\tan\theta d\theta$. Then

$$\begin{aligned}
\int \frac{dx}{x\sqrt{x^2-9}} &= \frac{1}{3}\int \frac{3\sec\theta\,\tan\theta}{3\sec\theta\sqrt{\sec^2\theta-1}}d\theta \\
&= \frac{1}{3}\int \frac{\tan\theta}{\sqrt{\tan^2\theta}}d\theta \\
&= \frac{1}{3}\int \frac{\tan\theta}{\pm\tan\theta}d\theta \\
&= \pm\frac{1}{3}\int d\theta = \pm\frac{1}{3}\theta + C \\
&= \pm\frac{1}{3}\text{arcsec}(\frac{\text{x}}{3}) + \text{C}
\end{aligned}$$

(3) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}}$
Solution: First we must complete the square.

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+2x+5}} &= \int \frac{dx}{\sqrt{(x^2+2x+1)+5-1}} \\ &= \int \frac{dx}{\sqrt{(x+1)^2+4}} = \frac{1}{2}\int \frac{dx}{\sqrt{(\frac{x+1}{2})^2+1}}\end{aligned}$$

Let $\frac{x+1}{2} = \tan\theta$, thus $x = 2\tan\theta - 1$ and $dx = 2\sec^2\theta d\theta$. Then

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+2x+5}} &= \frac{1}{2}\int \frac{2\sec^2\theta}{\sqrt{\tan^2\theta+1}} d\theta \\ &= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}} d\theta = \int \frac{\sec^2\theta}{|\sec\theta|} d\theta \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{\sqrt{(x+1)^2+4}}{2} + \frac{x+1}{2}\right| + C \\ &= \ln\left|\frac{\sqrt{(x+1)^2+4}+x+1}{2}\right| + C\end{aligned}$$

(4) $\displaystyle\int \frac{x}{\sqrt{4x-x^2}} dx$
Solution: First we must complete the square.

$$\begin{aligned}\int \frac{x}{\sqrt{4x-x^2}} dx &= \int \frac{x}{\sqrt{-(x^2-4x)}} dx = \int \frac{x}{\sqrt{4-(x^2-4x+4)}} dx \\ &= \int \frac{x}{\sqrt{4-(x-2)^2}} = \frac{1}{2}\int \frac{x}{\sqrt{1-(\frac{x-2}{2})^2}}\end{aligned}$$

Let $\frac{x-2}{2} = \sin\theta$, then $x = 2\sin\theta + 2$ and $dx = 2\cos\theta d\theta$. So,

$$\begin{aligned}\int \frac{x}{\sqrt{4x-x^2}} dx &= \frac{1}{2}\int \frac{2(\sin\theta+1)}{\sqrt{1-\sin^2\theta}} (2\cos\theta) d\theta \\ &= 2\int \frac{\sin\theta+1}{\sqrt{\cos^2\theta}} \cos\theta d\theta \\ &= 2\int (\sin\theta+1) d\theta = 2(-\cos\theta+\theta) + C \\ &= 2\left(-\frac{\sqrt{4-(x-2)^2}}{2} + \arcsin\left(\frac{x+2}{2}\right)\right) + C\end{aligned}$$