Worksheet #4

Perform the indicated integrations.

(1) $\displaystyle\int \frac{3}{x^2-1}dx$
Solution: First we do partial fractions.

$$\frac{3}{x^2-1} = \frac{3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{Ax+A+Bx-B}{x^2-1}$$

Matching coefficients we find $A+B=0$ and $A-B=3$, then $A=3/2$ and $B=-3/2$.

$$\begin{aligned}\int \frac{3}{x^2-1}dx &= \int \left(\frac{3}{2}\frac{1}{x-1} - \frac{3}{2}\frac{1}{x+1}\right)dx \\ &= \frac{3}{2}\left(\ln|x-1| - \ln|x+1|\right) + C \\ &= \frac{3}{2}\ln\left|\frac{x-1}{x+1}\right| + C\end{aligned}$$

(2) $\displaystyle\int \frac{2x^2-x-20}{x^2+x-6}dx$
Solution: By long division,

$$\frac{2x^2-x-20}{x^2+x-6} = 2 - \frac{3x+8}{x^2+x-6}.$$

By partial fractions

$$\frac{3x+8}{x^2+x-6} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{Ax-2A+Bx+3B}{x^2+x-6}.$$

Matching coefficents, we find $A+B=3$ and $-2A+3B=8$. Thus, $A=1/5$ and $B=14/5$. The integral becomes

$$\begin{aligned}\int \frac{2x^2-x-20}{x^2+x-6}dx &= \int \left(2 - \frac{1}{5}\frac{1}{x+3} - \frac{14}{5}\frac{1}{x-2}\right)dx \\ &= 2x - \frac{1}{5}\ln|x+3| - \frac{14}{5}\ln|x-2| + C\end{aligned}$$

(3) $\displaystyle\int \frac{x+1}{(x-3)^2}dx$
Solution: By partial fractions,

$$\frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{Ax-3A+B}{(x-3)^2}.$$

Matching coefficients, we find that $A = 1$ and $B = 1 + 3A = 4$.

$$\begin{aligned}\int \frac{x+1}{(x-3)^2}dx &= \int \frac{1}{x-3} + \frac{4}{(x-3)^2}dx \\ &= \ln|x-3| - 4\frac{1}{x-3} + C\end{aligned}$$

(4) $\displaystyle\int \frac{-4x^2+6x-39}{(2x-1)(x^2+9)}dx$
Solution: By partial fractions,

$$\frac{-4x^2+6x-39}{(2x-1)(x^2+9)} = \frac{A}{(2x-1)} + \frac{Bx+C}{x^2+9} = \frac{Ax^2+9A+(Bx+C)(2x-1)}{(2x-1)(x^2+9)}.$$

Matching coefficients, we find $A + 2B = -4$, $2C - B = 6$ and $9A - C = -39$, thus $A = -4$, $B = 0$ and $C = 3$.

$$\begin{aligned}\int \frac{-4x^2+6x-39}{(2x-1)(x^2+9)}dx &= \int \frac{-4}{(2x-1)} + \frac{3}{(x^2+9)}dx \\ &= -2\ln|2x-1| + 3\int \frac{1}{x^2+9}dx \\ &= -2\ln|2x-1| + 3\left(\frac{1}{3}\arctan\left(\frac{x}{3}\right)\right) + C \text{ by integral below}\end{aligned}$$

The last integral

$$\begin{aligned}\int \frac{1}{x^2+9}dx &= \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}dx \\ &= \frac{1}{9}\int \frac{3\sec^2\theta}{\tan^2\theta+1}d\theta \text{ via trig sub } \frac{\text{x}}{3} = \tan\theta \\ &= \frac{1}{3}\int \frac{\sec^2\theta}{\sec^2\theta}d\theta \\ &= \frac{1}{3}\int d\theta = \frac{1}{3}\theta + C \\ &= \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C\end{aligned}$$