Worksheet #5

(1) Determine if the sequence converges. If it does, find its limit.

(a) $a_n = \frac{3n+2}{n+1}$

Solution:

$$\lim_{n\to\infty} \frac{3n+2}{n+1} = \lim_{n\to\infty} \frac{3+\frac{2}{n}}{1+\frac{1}{n}} = 3$$

(b) $a_n = e^{-n}\sin n$

Solution: We know that $-e^{-n} \leq a_n \leq e^{-n}$, we also know $\lim_{n\to\infty} -e^{-n} = 0$ and $\lim_{n\to\infty} e^{-n} = 0$. Therefore by the *Squeeze Theorem*, $\lim_{n\to\infty} e^{-n}\sin n = 0$.

(c) $a_n = \frac{5n^3+2n+4}{n^2+6}$

Solution:

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{5n^3+2n+4}{n^2+6} \\ &= \lim_{n\to\infty} \frac{5n+\frac{2}{n}+\frac{4}{n^2}}{1+\frac{6}{n^2}} \\ &\to \infty\end{aligned}$$

Thus the sequence diverges.

(d) $a_n = \left(1+\frac{2}{n}\right)^{n/2}$

Solution: Since there is an n in the exponent, we need to think outside the box. Recall that $a_n = \left(1+\frac{2}{n}\right)^{n/2} = e^{\ln\left(\left(1+\frac{2}{n}\right)^{n/2}\right)}$. Thus we can move the limit to the exponent and work on it there.

$$\begin{aligned}\lim_{n\to\infty} \ln\left(\left(1+\frac{2}{n}\right)^{n/2}\right) &= \lim_{n\to\infty} \frac{n}{2}\ln\left(1+\frac{2}{n}\right) \\ &= \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{2}{n}}\end{aligned}$$

Both the numerator and the denominator go to 0 with n, therefore we can use L'Hopital's rule.

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{2}{n}} = \lim_{n\to\infty} \frac{\frac{-2}{n^2(1+\frac{2}{n})}}{-\frac{2}{n^2}} \qquad = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 1$$

Thus, $\lim_{n\to\infty} a_n = e^1 = e$.

(2) Indicate if the series converges. If it converges, find its sum.

(a) $\displaystyle\sum_{k=0}^{\infty} \left[2(\frac{1}{4})^k + 3(-\frac{1}{5})^k\right]$

Solution: Notice we are summing two geometric series where $r_1 = \frac{1}{4}$ and $r_2 = \frac{-1}{5}$.

$$\begin{aligned}\sum_{k=0}^{\infty}\left[2(\frac{1}{4})^k+3(-\frac{1}{5})^k\right] &= 2\sum_{k=0}^{\infty}(\frac{1}{4})^k+3\sum_{k=0}^{\infty}(-\frac{1}{5})^k \\ &= \frac{2}{1-\frac{1}{4}}+\frac{3}{1-(\frac{-1}{5})} \\ &= \frac{2}{\frac{3}{4}}+\frac{3}{\frac{6}{5}}\end{aligned}$$

(b) $\sum_{k=0}^{\infty}(\frac{9}{8})^k$

Solution: This series diverges.

(c) $\sum_{k=1}^{\infty}\frac{2}{(k+2)k}$

Solution: Notice that we can split the fraction up via partial fractions.

$$\frac{2}{(k+2)k}=\frac{A}{k}+\frac{B}{k+2}=\frac{Ak+2A+Bk}{k(k+2)}$$

Matching coefficients, we find that $A=1$ and $B=-1$. Thus,

$$\sum_{k=1}^{\infty}\frac{2}{(k+2)k}=\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+2}\right)$$

We look at the partial sums of this series.

$$S_1=1-\frac{1}{3}$$
$$S_2=1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}$$
$$S_3=1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}=1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}$$
$$S_4=1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}$$
$$\begin{aligned}S_N &= 1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\cdots-\frac{1}{N+2} \\ &= 1+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}\end{aligned}$$

Now we can take the limit.

$$\sum_{k=1}^{\infty}\frac{2}{(k+2)k}=\lim_{N\to}S_N=\frac{3}{2}$$