Math 8 Midterm 2 Solutions

November 8, 2011

(1) (10 pts) Let $f(x) = x^3 \ln(1+x^2)$. Find the Maclaurin series for f.
Solution: We know $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ for $|r| < 1$. Thus,

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x} dx \\
&= \int \sum_{n=0}^{\infty} (-x)^n dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n x^n dx \\
&= \sum_{n=0}^{\infty} (-1)^n \int x^n dx \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.
\end{aligned}$$

This means that

$$\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{2n+2}}{n+1}.$$

Finally, we find

$$x^3 \ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{2n+5}}{n+1}.$$

(2) (14 pts) Find the interval of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{5^n\sqrt{n}}$$

Solution: We shall use the ratio test.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{5^{n+1}\sqrt{n+1}}\frac{5^n\sqrt{n}}{(x-5)^n}\right| &= \lim_{n\to\infty}\left|\frac{(x-5)}{5}\sqrt{\frac{n}{n+1}}\right| \\ &= \frac{|x-5|}{5}\end{aligned}$$

So the series will converge when $\frac{|x-5|}{5} < 1$ or $0 < x < 10$. Now we must check the endpoints.

At $x = 0$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. This series converges by the alternating series test.

At $x = 10$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. This series diverges by the p-test.

Thus the interval of convergence is $0 \leq x < 10$.

(3) Consider the following series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^n}{n!}$$

(a) (4 pts) What function is the series equal to?
Solution:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = e^{-4x}$$

(b) (10 pts) Suppose the 4th degree Taylor polynomial, $T_4(x)$, is used to approximate the series for $|x| < 1$. Give a bound on the error in using this approximation.

Solution: We must use the Taylor inequality.

$$|R_4| \leq \frac{M}{5!}|x|^5 < \frac{M}{5!}$$

Now

$$M = \sup_{|x|\leq 1} |f^5(x)| = \sup_{|x|\leq 1} 4^5 e^{-4x} = 4^5 e^4$$

Thus

$$|R_4| < \frac{4^5 e^4}{5!}.$$

(4) Let O be the origin, P the point $(1, 2, 3)$, and Q the point $(0, -2, 2)$.

(a) (6 pts) Find the area of the parallelogram with P and Q the two vertices adjacent to vertex O.

Solution: $\vec{OP} =< 1, 2, 3 >$ and $\vec{OQ} =< 0, -2, 2 >$. We know that the area of the parallelogram determined by these vectors is given by

$$A = |\vec{OP} \times \vec{OQ}| = | < 10, 2, -2 > |\sqrt{108}$$

(b) (6 pts) Find an equation for the plane containing O, P, and Q.

Solution: To make a plane, we need a point and a normal vector. We computed the normal vector in the previous problem. $\vec{n} =< 10, 2, -2 >$ We shall use the origin as our point.

Thus the plane is given by

$$\vec{n} \cdot < x, y, z >= 10x + 2y - 2z = 0.$$

(5) Let a curve in 3-space be given by

$$\boldsymbol{r}(t) = \langle 4t,\ \sin(3t) + 2,\ \cos(3t) - 1\rangle.$$

(a) (6 pts) Find the tangent line at $t = 0$.
Solution: To make a line, we need a point and a direction. Our point is $\mathbf{r}(0) =< 0, 2, 0 >$. Our direction is $\mathbf{r}'(0)$.
Now, $\mathbf{r}'(t) =< 4, 3\cos(3t), 3\sin(3t) >$. Thus $\mathbf{r}'(0) =< 4, 3, 0 >$. So our line is determined by the parameterization

$$\begin{aligned} x &= 4t \\ y &= 2 + 3t \\ Z &= 0 \end{aligned}$$

(b) (6 pts) For what b is the length of the curve from $t = 0$ to $t = b$ equal to 10?

Solution: The length is given by

$$\begin{aligned} l &= \int_0^b |\mathbf{r}'(t)| dt \\ &= \int_0^b \sqrt{4 + 9(\cos^2(3t) + \sin^2(3t))} dt \\ &= \int_0^b \sqrt{5} dt \\ &= 5b \end{aligned}$$

We want $l = 10$. Thus $b = 2$.

(6) (12 pts) Consider again the curve in 3-space given by

$$\boldsymbol{r}(t) = \langle 4t,\ \sin(3t) + 2,\ \cos(3t) - 1\rangle.$$

Find the curvature.

Solution: Recall, that curvature is

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}$$

Previously, we found $\mathbf{r}'(t) =< 4, 3\cos(3t), -3\sin(3t) >$ and $|\mathbf{r}'(t)| = 5$. So $\mathbf{r}'$“$(t) =< 0, -9\sin(3t), -9\cos(3t) >$. Then

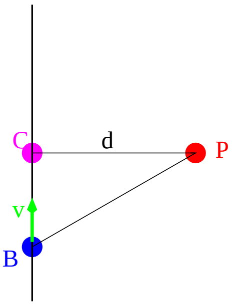
$$\kappa(t) = \frac{9\sqrt{2}}{5^2}.$$

(7) (10 pts) Find the distance between the line

$$x = t - 1, \quad y = 2t + 3, \quad z = 2 - t$$

and the point $(1, 3, -2)$.

Solution:



Let $P = (1, 3, -2)$, B the a point on the line corresponding to $t = 0$ (ie. $B = (-1, 3, 2)$) and $\mathbf{v}$ be the direction vector of the line $\mathbf{v} =< 1, 2, -1 >$.

Our goal is to find $d = |\vec{CP}|$. We know $|\vec{CB}| = \frac{\mathbf{v} \cdot \vec{BP}}{|\mathbf{v}|}$. Since $\vec{BP} =< -2, 0, 4 >$, $|\vec{CB}| = \frac{6}{\sqrt{6}}$. By Pythagorean thm,

$$d^2 = |\vec{BP}|^2 - |\vec{CB}|^2 = 20 - 6 = 14.$$

Thus the distance is $d = \sqrt{14}$.

(8) (8 pts) Find $\boldsymbol{r}(t)$ if $\boldsymbol{r}'(t) = \langle \cos(t), e^t, 2\rangle$ and $\boldsymbol{r}(0) = \langle 1, 1, 1\rangle$.
Solution:

$$\begin{aligned}\mathbf{r}(t) &= \int \langle \cos(t), e^t, 2\rangle dt \\ &= \langle \sin(t), e^t, 2t\rangle + \langle c_1, c_2, c_3\rangle\end{aligned}$$

Now $\boldsymbol{r}(0) = \langle 1, 1, 1\rangle$ which implies $\langle c_1, c_2, c_3\rangle =< 1, 0, 1 >$. Thus

$$\mathbf{r}(t) = \langle \sin(t) + 1, e^t, 2t + 1\rangle.$$

(9) (8 pts) Determine whether the following statements are true or false. You need not show your work and there will be no partial credit.

(a) $\underline{\quad F \quad}$ The only curves with constant curvature are straight lines and circles.

(b) $\underline{\quad F \quad}$ The lines $\langle 4t-1, 3-2t, 6t\rangle$ and $\langle 3-2t, t+1, 4-3t\rangle$ are skew.

(c) $\underline{\quad T \quad}$ The line $\langle 3t+2, t-1, 2t-5\rangle$ and the plane $x-y-z=0$ are parallel.

(d) $\underline{\quad F \quad}$ The angle between the planes $2x-y+3z=1$ and $x+y+z=20$ is $3\pi/4$.