1. Consider the function

\[ f(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)^2}. \]

For which values of \( x \) is \( f(x) \) defined?

**Solution:** The function \( f(x) \) is defined at values of \( x \) for which the series converges.

We find the radius of convergence using the ratio test.

\[
\lim_{n \to \infty} \left| \frac{(x-3)^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{(x-3)^n} \right| = |x-3| \lim_{n \to \infty} \left| \frac{(n+1)^2}{(n+2)^2} \right| = |x-3|.
\]

The ratio test tells us that the series converges when \( |x-3| < 1 \) and diverges when \( |x-3| > 1 \).

When \( |x-3| = 1 \) the ratio test fails, and we have to try something else. For \( x-3 = 1 \), or \( x = 4 \), we have

\[
\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2},
\]

which converges because it is a \( p \)-series with \( p = 2 \). For \( x-3 = -1 \), or \( x = 2 \), we have

\[
\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2},
\]

which converges by the alternating series test. It also converges because it converges absolutely.

Therefore the series converges, and \( f(x) \) is defined, when \( |x-3| \leq 1 \), or when \( 2 \leq x \leq 4 \).
2. (a) What is the Maclaurin series (the Taylor series about $x = 0$) for the function $f(x) = e^x$?

You do not need to show any work for this part of the problem, so if you remember the answer, you can just write it down.

**Solution:**

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!}. \]

(b) Find the Maclaurin series for $g(x) = e^{-x^2}$.

**Solution:** Substituting $-x^2$ for $x$ in the Maclaurin series for $e^x$, we get

\[ \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}. \]

(c) Let $F(x) = \int_0^x e^{-t^2} \, dt$. Find the first four nonzero terms of the Maclaurin series for $F(x)$.

**Solution:** Replacing $x$ by $t$ in the Maclaurin series for $e^{-x^2}$ yields the Maclaurin series for the integrand

\[ e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \]

Integrating this power series term-by-term

\[ \int e^{t^2} \, dt = C + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n + 1)n!} \]

Evaluating the integral from $t = 0$ to $t = x$ gives

\[ \int_0^x e^{t^2} \, dt = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n + 1)n!} \right]_{t=0}^{t=x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)n!} \]

\[ = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \]

So the first four nonzero terms are

\[ x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} \]
3. Find the Taylor series for $f(x) = \ln(x) - \ln(3)$ centered about $a = 3$.

**Solution:** Calculating the first few derivatives of $f(x)$, we have

\[
\begin{align*}
 f^{(0)}(x) &= \ln(x) - \ln(3) \\
 f^{(1)}(x) &= 1/x \\
 f^{(2)}(x) &= -1/x^2 \\
 f^{(3)}(x) &= 2/x^3 \\
 f^{(4)}(x) &= -2 \cdot 3/x^4 \\
 f^{(5)}(x) &= 2 \cdot 3 \cdot 4/x^4
\end{align*}
\]

we find that if $n \geq 1$, then the $n$-th derivative is $f^{(n)}(x) = (-1)^{n+1}(n - 1)!/x^n$. Hence Taylor’s formula gives

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

\[
= f(3) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

\[
= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n - 1)!}{3^n n!} (x - 3)^n
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 3^n} (x - 3)^n
\]
4. Suppose that $\vec{a}$ and $\vec{b}$ are two vectors in $\mathbb{R}^3$ such that if $\vec{a}$ and $\vec{b}$ are drawn emanating from the origin they both lie in the $xy$-plane, $\vec{a}$ in the third quadrant ($x < 0$ and $y < 0$) and $\vec{b}$ in the second quadrant ($x < 0$ and $y > 0$).

Suppose also that we know $|\vec{a}| = 1$ and $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 1$.

(a) Is the projection of $\vec{b}$ onto $\vec{a}$ longer than $\vec{a}$, shorter than $\vec{a}$, or the same length as $\vec{a}$?

**Solution:**

If $\theta$ is the angle between $\vec{a}$ and $\vec{b}$, then

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = 1 \quad |\vec{b}| = \sqrt{\vec{b} \cdot \vec{b}} = 2 \quad \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{2}$$

The length of the projection of $\vec{b}$ onto $\vec{a}$ is $|\vec{b}| |\cos \theta| = 1$, so it is the same length as $\vec{a}$.

(b) In what direction does $\vec{a} \times \vec{b}$ point?

**Solution:**

It must point in a direction normal to both $\vec{a}$ and $\vec{b}$, that is, normal to the $xy$-plane, so either the direction given by $\hat{k}$ (the positive $z$ direction) or the direction given by $-\hat{k}$ (the negative $z$ direction). Looking down from the top ($\hat{k}$, or positive $z$, direction) of the $xy$-plane we see that from $\vec{a}$ to $\vec{b}$ is a clockwise direction, so by the right-hand rule, the cross product points in the direction given by $-\hat{k}$.

(c) Find the length of $\vec{a} \times \vec{b}$.

**Solution:**

If $\cos \theta = \frac{1}{2}$ then $\sin \theta = \frac{\sqrt{3}}{2}$ (we know it cannot be negative because we always take $\theta$ to be an acute angle) so

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta = \sqrt{3}$$
5. Find an equation for the plane that contains the two lines $L_1$ and $L_2$:

$$L_1: \quad x = t + 2 \quad y = 3t - 5 \quad z = 5t + 1$$

$$L_2: \quad x = 5 - t \quad y = 3t - 10 \quad z = 9 - 2t$$

**Solution:** A vector that points in the same direction as $L_1$ is $v_1 = \langle 1, 3, 5 \rangle$ and one that points in the same direction as $L_2$ is $v_2 = \langle -1, 3, -2 \rangle$. Since the plane contains both lines, a normal vector to the plane must be orthogonal to $v_1$ and $v_2$. We can construct a normal vector using the cross product:

$$n = v_1 \times v_2 = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 5 \\ -1 & 3 & -2 \end{bmatrix} = (-6 - 15)\hat{i} - (-2 + 5)\hat{j} + (3 + 3)\hat{k} = \langle -21, -3, 6 \rangle$$

Since the plane contains both lines, we can find a point in the plane by finding a point on either line, using any parameter value of $t$. Taking $t = 0$ in the parametric equations for $L_1$, we know that $(2, -5, 1)$ is contained in the plane. So an equation for the plane is

$$\langle -21, -3, 6 \rangle \cdot \langle x - 2, y + 5, z - 1 \rangle = 0$$

or

$$-21x - 3y + 6z + 21 = 0.$$