

**Math 8**  
**Winter 2015**  
**Applications of Integration**

Here are a few important applications of integration.

The applications you may see on an exam in this course include only the Net Change Theorem (which is really just the Fundamental Theorem of Calculus, interpreted in some applied context), plus the ones we review in class, including average value, volumes by slicing, work as the integral of force, and possibly others later in the term. However, it's worth thinking about all of them, particularly for those of you who are taking calculus in order to use it in engineering or some other subject.

**The Net Change Theorem** (for functions of time): This is just the Fundamental Theorem of Calculus, applied. The Fundamental Theorem of Calculus tells us that, if  $F$  has a continuous derivative on the interval  $[a, b]$ , then

$$\int_a^b F'(t) dt = F(b) - F(a).$$

If  $t$  represents time, and  $F(t)$  represents some quantity that changes over time, then  $F'(t)$  represents the rate of change of  $F(t)$  (with respect to time), and  $F(b) - F(a)$  represents the *net change* in  $F(t)$  between times  $t = a$  and  $t = b$ .

(“Net change” means that we allow decreases and increases to cancel each other out. For example, if the temperature is  $-15$  degrees at 7 AM, rises to 13 degrees at 2 PM, and then drops to 11 degrees at 7 PM, then between 7 AM and 7 PM the temperature rises by 28 degrees and then falls by 2 degrees, and the net change is 26 degrees; a positive net change denotes an increase.<sup>1</sup>)

This is what the Net Change Theorem says. Quoting directly from our textbook:

Net Change Theorem. The integral of a rate of change is the net change:

$$\boxed{\int_a^b F'(x) dx = F(b) - F(a)}.$$

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<sup>1</sup>I got these numbers from the Weather Underground forecast for January 8, 2015 in Hanover, New Hampshire; degrees are in Fahrenheit.

Pages 324 and 325 of the textbook list several instances of the Net Change Theorem.

In particular, the Net Change Theorem tells us that the integral of velocity is distance: For an object traveling along a straight path (with positive and negative directions chosen), if  $F(t)$  denotes the distance from the starting point at time  $t$ , then  $F'(t)$  denotes velocity, the rate of change of distance with respect to time, and the integral of the velocity between times  $t = a$  and  $t = b$

$$\int_a^b F'(t) dt$$

denotes the net distance traveled between times  $t = a$  and  $t = b$ .

**Average Value:** The average value of a function  $f(x)$  for  $x$  in the interval  $[a, b]$  is given by

$$\boxed{\frac{1}{b-a} \int_a^b f(x) dx}.$$

This is very like the formula for computing the average of finitely many numbers: To find the average of a finite set of numbers, add up the numbers, and divide by the size of the set. To find the average value of a function  $f(x)$  on an interval, use an integral to “total up” the values of  $f$ , and divide by the length of the interval.

We could approach this more formally: To approximate the average value of  $f(x)$  on the interval  $[a, b]$ , divide the interval  $[a, b]$  into a large number of equal subintervals (say  $n$  subintervals), choose a point  $x_i$  in the  $i^{\text{th}}$  subinterval for  $i = 1, 2, \dots, n$ , and take the average of the values  $f(x_i)$ . This gives us

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n f(x_i) &= \frac{1}{b-a} \frac{b-a}{n} \sum_{i=1}^n f(x_i) = \frac{1}{b-a} \sum_{i=1}^n f(x_i) \frac{b-a}{n} = \\ &= \frac{1}{b-a} \sum_{i=1}^n f(x_i) \Delta x, \end{aligned}$$

where  $\Delta x = \frac{b-a}{n}$  is the size of the subintervals. This sum is just the “Riemann sum” that we used to approximate the area under a curve. When we compute the average as the limit of closer and closer approximations as

$n \rightarrow \infty$ , we get

$$\frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \frac{1}{b-a} \int_a^b f(x) dx.$$

**Note:** Formal computations, such as the derivation of the formula for average value above, are not of interest only to theoretical mathematicians. If you are using calculus in engineering or economics or medical research, you don't want to be limited to the applications of integration on some list; you want to know how to recognize a new application of integration "in the wild," so to speak. You recognize a potential application of integration by seeing that you can approximate the thing you're interested in as a sum of small pieces, and get a better approximation by using a larger number of smaller pieces. Then, some computation like this is needed to tell you exactly what integral you should use to compute that thing you're interested in.

In this course, we won't expect you to derive these formulas, such as this formula for average value, just to apply them. However, in a physics or engineering course, you may well have to do problems such as some of the computations of work in the section on "other applications," that essentially require you to figure out the appropriate integral in this way.

The following sections will sometimes give these formal computations, and sometimes give more intuitive arguments. You might try to provide the formal computation when it is not included.

**Volumes by Slicing:** To find the volume of a solid object occupying the region in space between  $x = a$  and  $x = b$ , where  $x$  is measured along some straight axis, integrate the object's cross-sectional area between  $x = a$  and  $x = b$ . More precisely, suppose  $A(x)$  is the cross-sectional area at  $x$ ; that is, the cross-sectional area we get if we slice the object at the point  $x$ , perpendicularly to the axis. Then the volume of the object is given by

$$V = \int_a^b A(x) dx.$$

We can see why this is the case just as we did with the formula for average value: We can approximate the volume of the object by cutting it, perpendicular to the  $x$ -axis, into many thin slices of thickness  $\Delta x$ . The volume of each thin slice can be approximated by the cross-sectional area

$A(x_i)$  at some particular point, times the thickness  $\Delta x$  of the slice. Then we can approximate the volume of the object by adding up these approximate volumes  $A(x_i)\Delta x$  of the slices, and find the exact volume by taking the limit as the number of slices approaches infinity.

We can also give an informal argument that this makes sense. If the cross-sectional area is the same at any point, the volume of the object is the product of its length and its cross-sectional area. For example, a cylinder of length  $h$  and cross-section a circle of radius  $r$  has volume  $V = \pi r^2 h$ , a formula that may be familiar. It makes sense that if the cross-sectional area varies, you could possibly get the volume by multiplying the length by the average cross-sectional area. Using the formula for average value, we get

$$V = (b - a) \frac{1}{b - a} \int_a^b A(x) dx = \int_a^b A(x) dx.$$

**Work:** In physics, if a force of magnitude  $F$  acts on an object as it moves a distance  $d$  (where the force is acting in the direction of the object's motion), the work done by that force on that object is the product  $Fd$ . Informally, we say work equals force times distance.

If the force is negative, that indicates that the force is acting opposite to the direction of motion, and the work  $Fd$  is also negative. Intuitively, positive work is work done by a force acting in the direction of motion, and negative work indicates that work must be done to oppose a force acting opposite to the direction of motion.

If the force acting on the object is different at different points along its path, then we compute work using an integral: Suppose an object moves along the  $x$ -axis from point  $x = a$  to point  $x = b$ , and a force on the object acting in the  $x$ -direction has magnitude  $F(x)$  at point  $x$ . (Again, if  $F(x)$  is negative, that indicates that the force is acting in the negative  $x$ -direction.) Then the net work done by that force on the object is given by the integral of  $F(x)$  over the interval of motion:

$$W = \int_a^b F(x) dx.$$

To see why this is the case, consider approximating the work: Divide the interval of motion  $[a, b]$  into a large number  $n$  of equal subintervals of length

$\Delta x = \frac{b-a}{n}$ . If  $n$  is large enough,  $\Delta x$  will be small enough that the force doesn't change very much over any one subinterval, and we can approximate the force in the  $i^{\text{th}}$  subinterval by the force  $F(x_i)$  at any one point  $x_i$  in that subinterval. Then, because work equals force times distance, the work done as the object moves along the  $i^{\text{th}}$  subinterval is approximately  $F(x_i)\Delta x$ .

We approximate the work over the entire interval by adding up the approximate values of the work over the subintervals:

$$W \approx \sum_{i=1}^n F(x_i) \Delta x.$$

We can get a closer approximation by taking a larger number of subintervals, and an exact value by taking the limit as  $n$  approaches infinity:

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n F(x_i) \Delta x.$$

We can recognize this limit as the definition of the definite integral:

$$W = \int_a^b F(x) dx.$$

Just as with volumes by slicing, we could also give an informal argument that this makes sense, because this is the distance times the average force. However, our intuition can mislead us — in real life, all too many things that make sense turn out not to be true — and therefore the reasoning with approximations and limits is not only more formal but also more reliable.

**Units of Work:** We write other quantities in terms of the basic quantities of mass, distance, and time. Newton's Second Law (force equals mass times acceleration) tells us the units of force should be the units of mass times the units of acceleration. In the SI system, force is measured in Newtons ( $N$ ), defined by

$$1N = 1kg\ m/s^2,$$

where  $kg$  is kilogram,  $m$  is meter, and  $s$  is second, and work is measured in Joules ( $J$ ), defined by

$$1J = 1kg\ m^2/s^2.$$

**Other Applications:** Another standard application is finding total mass by integrating mass density, total charge by integrating charge density, and so forth. For example, suppose a wire occupying the portion of the  $x$ -axis  $a \leq x \leq b$  has a variable mass density of  $\rho(x)$  grams per meter at point  $x$ . If the mass density were constant, we would just multiply the mass density (in grams per meter) by the total length (in meters) to get the total mass (in grams). Using an intuitive argument, we can compute the total mass as the length times the average mass density, or

$$(b - a) \frac{1}{b - a} \int_a^b \rho(x) dx = \int_a^b \rho(x) dx.$$

Computing work via an integral is often not as straightforward as integrating force over some interval of distance. For example, suppose a chain of length  $\ell$  and constant mass density  $\rho$  is hanging by one end from the top of a wall, and we want to compute the work required to pull the entire chain up to the top of the wall. The force of gravity on an object of mass  $m$  is  $mg$ , and this is the force we must do work against. More work is required to pull the last part of the chain up than the first part of the chain, since the last part of the chain must cover a longer distance.

Letting  $x$  be the distance from the top of the wall, the chain occupies the interval  $0 \leq x \leq \ell$ . We divide the interval into  $n$  subintervals of length  $\Delta x = \frac{b - a}{n}$ . Choose  $x_i$  in the  $i^{\text{th}}$  subinterval. The portion of the chain occupying the  $i^{\text{th}}$  subinterval has mass  $\rho \Delta x$ , and must be moved a distance of approximately  $x_i$ , so the work done on this portion of the chain is approximately the force on that portion of the chain ( $\rho(\Delta x)g$ ) times the approximate distance it must be moved,  $x_i$ , or  $\rho g x_i \Delta x$ . We approximate the work on the entire chain by adding up the work done on each small subinterval, and find the actual value by taking a limit as  $n$  approaches infinity:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \rho g x_i \Delta x = \int_0^{\ell} \rho g x dx.$$

You can find other examples of computing work, and other applications of integration, in the textbook.