Daily HW #3 Solutions (MK)

1. a) Approximate $f(x) = x^{2/3}$ at $a=1$ with $T_3(x)$.

$$f'(x) = \frac{2}{3}x^{-1/3} \qquad f'(1) = \frac{2}{3}$$

$$f''(x) = -\frac{2}{9}x^{-4/3} \qquad f''(1) = -\frac{2}{9}$$

$$f'''(x) = +\frac{8}{27}x^{-7/3} \qquad f'''(1) = \frac{8}{27}$$

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f'''(1)(x-1)^3$$

$$\boxed{T_3(x) = 1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3}$$

b) What is the error for $.8 \le x \le 1.2$?

To determine M:

Find extremes for $f^{(4)}(x) = -\frac{56}{81}x^{-10/3}$ in the interval $.8 \le x \le 1.2$.

$f^{(5)}(x) = \frac{560}{483}x^{-13/3}$, which is always positive in this interval,

So largest $|f^{(4)}(x)|$ occurs at left endpoint $x = .8 = \frac{4}{5}$.

We choose $M \ge |f^{(4)}(.8)|$

$$= \left|-\frac{56}{81}\left(\frac{4}{5}\right)^{-10/3}\right|$$

$$= \cancel{0.74474...}\ 1.4545...$$

$\boxed{M = \frac{3}{2}\ \cancel{\tfrac{3}{4}} = 1.5\ \cancel{0.75}}$ is a convenient upper bound.

Since $a=1$, we calculate bounds for $x-a = x-1$:

$$.8 - 1 \le x - 1 \le 1.2 - 1$$

$$-0.2 \le x-1 \le +0.2$$

$$\boxed{|x-1| \le 0.2}$$

Now we have

$$|R_3(x)| \le \frac{M\,|x-1|^4}{4!}$$

$$= \frac{3}{2}\,\frac{(0.2)^4}{24} = \frac{\cancel{3}}{2}\,\frac{1}{5^4}\,\frac{1}{2\cdot\cancel{3}\cdot 4}$$

$$= \frac{1}{2^4\cdot 5^4} = \frac{1}{10^4} = 0.0001.$$

$\boxed{\text{The error bound is } 0.0001.}$

2. How many terms should be used to estimate $e^{.01}$ to within $.0001$?

We use $f(x) = e^x$, and a convenient a close to $x = .01$ is $a = 0$.

Since $f^{(n)}(x) = e^x$ for any n, $|f^{(n)}(x)| = |e^x| = e^{0.0}$ for $x \in [0, 0.01]$ (we use that e^x is increasing here.)

Thus, $$|R_n(.01)| \leq \frac{e^{.01}|.01-0|^{n+1}}{(n+1)!} = \frac{e^{.01}(.01)^{n+1}}{(n+1)!},$$

and we need to choose n so that the right side is less than $.0001$:

$$\frac{e^{.01}(.01)^{n+1}}{(n+1)!} < .0001.$$

We start checking successive values of n:

$n = 1$: $0.0000505\ldots$

$n = 2$: $0.000000168\ldots$ ← This one is less than $.0001$.

So $\boxed{\text{two terms is enough.}}$

3. For What range of x-values is ~~can~~ $x - \frac{x^3}{6}$ an approximation of $\sin(x)$ with error less than .01?

Since $x - \frac{x^3}{6}$ is a degree 3 polynomial, it looks like it is $T_3(x)$ for some a. (In fact, it is also $T_4(x)$, but we won't use this.)

The fact that it is written as powers of x ~~$(x-a)$~~ ~~with $a=0$,~~ instead of $x-a$, leads us to guess that $a=0$. Direct calculation confirms this:

$$T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3$$

$$= \sin(0) + \cos(0)\cdot x + \frac{(-\sin(0))}{2}x^2 + \frac{(-\cos(0))}{6}x^3$$

$$= 0 + 1\cdot x - \frac{0}{2}x^2 - \frac{1}{6}x^3$$

$$= x - \frac{x^3}{6} \quad \checkmark$$

~~Since~~ Now we need to bound $|R_3(x)|$.

For M, we need $|f^{(4)}(x)| = |\sin(x)|$, but since the range of x-values is not determined yet, the best we can do is $|\sin(x)| \leq 1$, so we take $M=1$.

Then $|R_3(x)| \leq \frac{1\cdot|x-0|^4}{4!} = \frac{|x|^4}{24}$. We need the right side less than .01, so we solve for $\frac{|x|^4}{24} < .01$, to get $|x| < (.24)^{1/4} = .699927\ldots$, or $\boxed{-.69927 < x < .69927}$.