Math 8: Calculus of Functions of One and Several Variables
Midterm 1
Thursday, January 26

Name: Answer Key

Circle your section number: 1–Kobayashi 2–DeFord

Please read the following instructions before starting the exam:

- This exam is closed book, with no calculators, notes, or books allowed. You may not give or receive any help during the exam, though you may ask instructors for clarification if necessary.
- Be sure to **show all work** whenever possible. Even if your final answer is incorrect, we can assign an appropriate amount of partial credit if we can see how you arrived at your answer.
- Please circle or otherwise indicate your final answer if possible.
- The test has a total of 9 questions, worth a total of 90 points. Point values are indicated for each question.
- You will have two hours from the start of the exam to complete it.
- Good luck!

HONOR STATEMENT: I have neither given nor received help on this exam, and I attest that all the answers are my own work.

Signature: ______________________________

This page for grading purposes only.

Problem	Points	Scores
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	5	
8	5	
9	10	
Total	80	

1. **(10 points)** Determine whether the following series converge and explain your reasoning:

a) **(5 points)**

$$\sum_{n=1}^{\infty}\left(\frac{3n-1}{2n+4}\right)^n$$

Root test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{3n-1}{2n+4}\right)^n} = \lim_{n\to\infty}\frac{3n-1}{2n+4} = \frac{3}{2} > 1$$

Diverges

b) **(5 points)**

$$\sum_{n=1}^{\infty}\frac{2}{\sqrt[3]{6n^4-3}}.$$

Limit comparison test with $b_n = \frac{1}{n^{4/3}}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2}{\sqrt[3]{6n^4-3}}\cdot\frac{n^{4/3}}{1} = \lim_{n\to\infty}\frac{2}{\sqrt[3]{6-\frac{3}{n^{4/3}}}} = \frac{2}{\sqrt[3]{6}}$$

Thus, Σa_n and Σb_n have the same convergence behavior, we know Σb_n is a p-series with $p = 4/3 > 1$ so both series converge.

2. **(10 points)** Determine whether the following series converge, and explain the reason. If a series converges, compute its sum.

a) **(3 points)**

$$\sum_{n=0}^{\infty} e^n$$

- converge/diverge: Diverge
- reason: geometric series with $r > 1$
- sum (if it converges): N/A

b) **(3 points)**

$$\sum_{n=0}^{\infty} (-e)^{1-n} = (-e)\sum_{n=0}^{\infty}\left(-\frac{1}{e}\right)^n$$

- converge/diverge: Converge
- reason: Geometric series with $r < 1$
  or alternating series.
- sum (if it converges): $\frac{-e^2}{e+1}$

c) **(4 points)**

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{5^i} = \lim_{n\to\infty}\left(\frac{3}{5}\sum_{i=0}^{\infty}\frac{1}{5^i}\right)$$

- converge/diverge: Converge
- reason: Limit of partial sums of the geometric series $\sum \frac{1}{5^i}$
- sum (if it converges): $3/4$

3. **(10 points)** Determine if the following series converge, and explain your reasoning.

a) **(3 points)**

$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$$

- converge/diverge: Diverge
- reason: P-Series with $P = \frac{1}{2} < 1$

b) **(3 points)**

$$\sum_{n=1}^{\infty} \frac{2n}{(n^2+4)^3}$$

- converge/diverge: Converges
- reason: Integral test:

$$\int \frac{2x\,dx}{(x^2+4)^3} = \int \frac{du}{u^3} = -\frac{1}{u^2} = \frac{-1}{(x^2+4)^2}\Big|_1^{\infty} = \frac{1}{25}$$

$u = x^2+4$
$du = 2x\,dx$

Converges.

(4 points) Use the remainder estimate for the integral test to bound the error in approximating the following series with 9 terms:

$$\sum_{n=1}^{\infty} n^{-2}$$

$$\int_{10}^{\infty} \frac{1}{x^2}\,dx \leq R_9(x) \leq \int_9^{\infty} \frac{1}{x^2}\,dx$$

$$\frac{-1}{x}\Big|_{10}^{\infty} \leq R_9(x) \leq \frac{-1}{x}\Big|_9^{\infty}$$

$$\frac{1}{10} \leq R_9(x) \leq \frac{1}{9}$$

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4. **(10 points)** Determine if the following series are absolutely convergent, conditionally convergent, or divergent, and justify your work. If the series converges estimate the error in the approximation with 8 terms using the remainder estimate for alternating series. The convergent series is worth 6 points and the divergent series is worth 4 points.

a) **(? points)**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Alternating Series test

$|a_{n+1}| \leq |a_n|$ $\qquad \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ ✓

$\lim_{n\to\infty} |a_n| = 0$ $\qquad \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

converges. However $\sum |a_n| = \sum \frac{1}{\sqrt{n}}$ diverges p-series $P = \frac{1}{2} < 1$

So conditionally convergent.

Error estimate: $R_8(x) \leq |a_9| = \frac{1}{3}$

b) **(? points)**

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+4}{(3n+1)}$$

Alternating series test

$\lim_{n\to\infty} \frac{n^2+4}{3n+1} = \infty$ $\quad \neq 0$

Diverges

5. **(10 points)** Fill in the small blanks with the appropriate possibilities shown here and fill in the large blanks with an appropriate mathematical term:

- a_n
- $|a_n|$
- $(-1)^n a_n$
- $(-1)^n |a_n|$

a) **(2 points)** Rearrangements can affect the sum of a/an Conditionally convergent series.

b) **(3 points)** If $|a_n|$ is nonzero, then $\sum (-1)^n |a_n|$ is an alternating series.

c) **(5 points)** Definition: $\sum a_n$ is a Conditionally convergent series if $\sum a_n$ converges and $\sum |a_n|$ diverges.

6. **(10 points)**

a) **(7 points)** Determine the interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-11)^n}{n^2}$$

Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(x-11)^{n+1}}{(n+1)^2}}{\frac{(x-11)^n}{n^2}} \right| = \lim_{n\to\infty} (x-11)\cdot\left(\frac{n}{n+1}\right)^2 = x-11$$

To converge we need $|x-11| < 1 \rightarrow 10 < x < 12$

Check endpoints:

$\sum_{n=1}^{\infty} \frac{(10-11)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges - Alternating

$\sum_{n=1}^{\infty} \frac{(12-11)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges P-Series

Interval $[10, 12]$

b) **(3 points)** List the possible intervals for a power series using endpoints $a, b, \infty, -\infty$ where appropriate.

- $x \in (-\infty, \infty)$
- $x = \{a\}$
- $|x-a| < R$ + endpoints
  - (a, b)
  - $[a, b]$
  - $[a, b)$
  - $(a, b]$

7. **(5 points)** Find a power series centered at $a = 0$ and interval of convergence for $f(x) = \dfrac{2}{3-x}$.

$$\frac{2}{3-X} = \frac{2}{3}\left(\frac{1}{1-\frac{x}{3}}\right) = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$$

This is a geometric series so it converges where $\left|\frac{x}{3}\right| < 1$ i.e. $(-3,3)$

8. **(5 points)** Compute the third order Taylor polynomial for $\dfrac{2}{x}$ at $a = 1$.

$f(x) = \frac{2}{x}$ $\quad f(1) = 2$

$f'(x) = \frac{-2}{x^2}$ $\quad f'(1) = -2$

$f''(x) = \frac{4}{x^3}$ $\quad f''(1) = 4$

$f'''(x) = \frac{-12}{x^4}$ $\quad f'''(1) = -12$

$$T_3(x) = 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3$$

9. **(10 points)**

a) **(4 points)** Write out the general Taylor remainder inequality.

$$|R_n(x)| \leq \frac{M}{(n+1)^2} |x-a|^{n+1}$$

b) **(6 points)** Use the Taylor remainder inequality to bound the approximation $e^x \approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3$ on the interval $[-1, 3]$.

$n = 3$

$a = 1$

$d = 2$

$f''''(x) = e^x$ increasing so

$M = e^3$

max of $|x-a|$ is 2 on this interval

$$|R_3(x)| \leq \frac{e^3}{24} \cdot 2^4$$