Math 8: Calculus of Functions of One and Several Variables
Midterm 2
Thursday, February 16

Name: Answer Key

Circle your section number: 1–Kobayashi 2–DeFord

Please read the following instructions before starting the exam:

- This exam is closed book, with no calculators, notes, or books allowed. You may not give or receive any help during the exam, though you may ask instructors for clarification if necessary.
- Be sure to **show all work** whenever possible. Even if your final answer is incorrect, we can assign an appropriate amount of partial credit if we can see how you arrived at your answer.
- Please circle or otherwise indicate your final answer if possible.
- The test has a total of 18 questions, worth a total of 160 points. Point values are indicated for each question.
- You will have two hours from the start of the exam to complete it.
- Good luck!

HONOR STATEMENT: I have neither given nor received help on this exam, and I attest that all the answers are my own work.

Signature: ______________________

This page for grading purposes only.

Problem	Points	Scores	Problem	Points	Scores
1	10		10	5	
2	5		11	5	
3	5		12	10	
4	10		13	10	
5	10		14	10	
6	10		15	10	
7	10		16	10	
8	10		17	10	
9	10		18	10	
Total:			/160		

1. **(10 points)**

(a) **(2 points)** What are the initial and terminal points of the vector $\langle 1, 2, 3\rangle$ when its initial end is placed on the curve $\vec{r}(t) = \langle 2t, \ln(t-2), t^2 - 4\rangle$ at $t = 3$?

Initial point $r(3) = (6, 0, 5)$

Terminal Point $r(3) + \langle 1, 2, 3\rangle = (7, 2, 8)$

(b) **(2 points)** Find a unit vector pointing in the same direction as $\langle 5, 3, \sqrt{2}\rangle$.

$|\langle 5, 3, \sqrt{2}\rangle| = \sqrt{25+9+2} = 6$

unit vector $\langle 5/6, \frac{1}{2}, \frac{\sqrt{2}}{6}\rangle$

(c) **(6 points)** Find a vector with integer entries that is parallel to the tangent line of the graph of $\sin(x) + (4x+2)^2$ at $x = 0$.

$f(x) = \sin(x) + (4x+2)^2$

$f'(x) = \cos(x) + 2(4x+2) \cdot 4$

Slope $= f'(0) = 1 + 16 = 17$

$\langle 1, 17\rangle$

2. **(5 points)**

(a) **(3 points)** Use vectors to decide if the triangle with vertices $P = (1,1,1)$, $Q = (2,2,2)$, and $R = (-1,3,5)$ is a right triangle.

$a = Q - P = \langle 1,1,1 \rangle$

$b = Q - R = \langle 3,-1,-3 \rangle$

$c = R - P = \langle -2,2,4 \rangle$

$$\left.\begin{array}{l} a \cdot b = -1 \\ a \cdot c = 4 \\ b \cdot c = -20 \end{array}\right\} \text{ all non-zero so no right angles}$$

(b) **(2 points)** Use the cross product to find the area of the triangle in part (a).

$$\frac{|a \times c|}{2} = \left|\begin{matrix} i & j & k \\ 1 & 1 & 1 \\ -2 & 2 & 4 \end{matrix}\right| / 2 = \frac{|\langle 2,-6,4 \rangle|}{2} = \frac{\sqrt{4+36+16}}{2} = \sqrt{14}$$

3. **(5 points)** Let $\vec{a} = \langle 1,2,2 \rangle$ and $\vec{b} = \langle 3,-4,0 \rangle$. Find the following:

(a) $3\vec{a} - 2\vec{b}$ $\quad \langle 3,6,6 \rangle - \langle 6,-8,0 \rangle = \langle -3,14,6 \rangle$

(b) $\text{comp}_{\vec{a}} \vec{b}$

$$\frac{a \cdot b}{|a|} = \frac{3-8+0}{\sqrt{1+2^2+2^2}} = \frac{-5}{3}$$

(c) $\text{proj}_{\vec{b}} \vec{a}$

$$\left(\frac{a \cdot b}{|b|^2}\right) b = \frac{-5}{3^2+4^2} \langle 3,-4,0 \rangle = \langle -\tfrac{3}{5}, \tfrac{4}{5}, 0 \rangle$$

4. **(10 points)** Find the acute angle between the lines: $3x - y + 4 = 0$ and $-8x + 3y = 2$.

Slope of line 1 is 3 $\langle 1,3 \rangle = a$

Slope of line 2 is $8/3$ $\langle 1, \frac{8}{3} \rangle = b$

$\Theta = \cos^{-1}\left(\frac{a \cdot b}{|a| \cdot |b|}\right) = \cos^{-1}\left(\frac{9}{\sqrt{10}\sqrt{\frac{73}{9}}}\right) = \cos^{-1}\left(\frac{27}{\sqrt{730}}\right)$

5. **(10 points)** Determine if each of the pairs below is parallel, skew, or intersecting:

(a) **(2 points)** The planes $x - 2y + 3z = 17$ and $4x + 2y + 8z = 23$.

normal vectors $\langle 1,-2,3 \rangle$ $\langle 4,2,8 \rangle$ intersecting

(b) **(2 points)** The planes $3x - 4y + 5z = 6$ and $-9x + 12y - 15z = 0$.

normal vectors $\langle 3,-4,5 \rangle$ $\langle -9,12,-15 \rangle$ Parallel

(c) **(2 points)** The lines $\langle -5t - 1, 10t + 2, 35t - 4 \rangle$ and $\langle t + 1, -2t, -7t \rangle$.

direction vectors $\langle -5,10,35 \rangle$ $\langle 1,-2,-7 \rangle$ parallel

(d) **(2 points)** The lines $\langle 0, 2t, -t - 3 \rangle$ and $\langle 1, 4t, t + 3 \rangle$.

Skew

(e) **(2 points)** The lines $\langle t - 3, 2t - 6, -t + 7 \rangle$ and $\langle 3t - 11, -2t + 10, t - 1 \rangle$.

$t-3 = 3s-11$

$2t-6 = -2s+10$

$3t-9 = s-1$

$3t-8 = s$

$t-3 = 9t-35$

$-8t = -32$

$t = 4 \rightarrow s = 4$

at $t=4$ $\langle 1,2,3 \rangle$

at $s=4$ $\langle 1,2,3 \rangle$

intersecting

6. **(10 points)**

(a) **(2 points)** Give an example of two vectors where $\text{comp}_{\vec{a}}\vec{b} = \text{comp}_{\vec{b}}\vec{a}$.

$a = \langle 1, 2, 3 \rangle \qquad b = \langle 3, 2, 1 \rangle$

(b) **(2 points)** Give an example of two vectors where $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$.

$a = \langle 1, 2, 3 \rangle \qquad b = \langle -1, -2, -3 \rangle$

(c) **(4 points)** Determine if $\vec{a} = \langle 1, 0, 1 \rangle$, $\vec{b} = \langle -1, 2, 0 \rangle$, and $\vec{c} = \langle 3, 2, 1 \rangle$ are coplanar.

Scalar triple product!

$$\begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 3 & 2 & 1 \end{vmatrix} = 2 - 0 + (-8) = -6 \neq 0$$

not coplanar

(d) **(2 points)** If $\vec{a}$ and $\vec{b}$ are unit vectors what are the largest and smallest possible values of $|\vec{a} \times \vec{b}|$? What about $\vec{a} \cdot \vec{b}$?

Largest: $|\vec{a} \times \vec{b}|$ __1__ $\qquad \vec{a} \cdot \vec{b}$ __1__

Smallest: $|\vec{a} \times \vec{b}|$ __0__ $\qquad \vec{a} \cdot \vec{b}$ __-1__

7. **(10 points)** Determine which of the following expressions are meaningful where $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ are all 3–vectors. If not, explain why. If so, is the output a vector or a scalar?

(i) $(\vec{a} \cdot \vec{b}) \times \vec{c}$
Meaningful: NO
Output/explanation: Cannot cross a scalar and a vector

(ii) $\vec{a} \cdot (\vec{b} \times \vec{c})$
Meaningful: Yes
Output/explanation: Scalar

(iii) $((\vec{a} \times \vec{b}) \times \vec{c}) \cdot \vec{d}$
Meaningful: Yes
Output/explanation: Scalar

(iv) $(\vec{a} \cdot \vec{b}) \cdot (\vec{c} \times \vec{d})$
Meaningful: NO
Output/explanation: Cannot dot a scalar and a vector

(v) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$
Meaningful: Yes
Output/explanation: Scalar

8. **(10 points)** Compute the parametric and symmetric equations for the line through the points $(2, 5, -7)$ and $(-3, 1, 2)$.

$V = \langle -5, -4, +9 \rangle$

$\langle 2 - 5t, 5 - 4t, -7 + 9t \rangle$

$$\frac{x-2}{-5} = \frac{y-5}{-4} = \frac{z+7}{9}$$

9. **(10 points)** Find an equation of the plane through the points $(0, 0, 1)$, $(1, 1, 0)$, and $(1, 0, 0)$.

$v = \langle 1, 1, -1 \rangle$

$w = \langle 1, 0, -1 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & k \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{vmatrix} = \langle -1, 0, -1 \rangle$$

$\langle 1, 0, 1 \rangle \cdot \langle x - 0, y - 0, z - 1 \rangle = 0$

$x + z - 1 = 0$

10. **(5 points)** Find the distance of the point $(1, 2, 3)$ from the plane described by $-2x + 6y - z = 4$.

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

$$\frac{|-2+12-3-4|}{\sqrt{4+36+1}} = \frac{3}{\sqrt{41}}$$

11. **(5 points)** Find the distance between the parallel planes $x + 2y - z + 7 = 0$ and $3x + 6y - 3z = 0$.

Point $(0,0,7)$ lies on the first plane

$$\frac{|0(3)+(0)\cdot 6+7(-3)+0|}{\sqrt{9+36+9}} = \frac{21}{\sqrt{54}} = \frac{7}{\sqrt{6}}$$

12. **(10 points)**

(a) **(3 points)** Find the angle between the intersecting planes $-x+3y-2z+1 = 0$ and $x+6y-z+2=0$.

Same as angle between normal vectors

$$\theta = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1| \cdot |n_2|}\right) = \cos^{-1}\left(\frac{\langle -1, 3, -2\rangle \cdot \langle 1, 6, -1\rangle}{\sqrt{14} \cdot \sqrt{38}}\right)$$

$$= \cos^{-1}\left(\frac{19}{\sqrt{14} \cdot \sqrt{38}}\right)$$

(b) **(7 points)** Find the equation of the line that lies at the intersection of these planes. Set $y=0$ to find point:

$$-x-2z=-1$$
$$x-z=-2$$
$$-3z=-3$$
$$z=1$$
$$x=-1$$

Point $\theta = (-1, 0, 1)$

direction vector is the cross product of normal vectors:

$$\begin{vmatrix} i & j & k \\ -1 & 3 & -2 \\ 1 & 6 & -1 \end{vmatrix} = \langle 9, -3, -9\rangle$$

line $\langle -1+9t, -3t, 1-9t\rangle$

13. **(10 points)** For each equation below, determine the type of surface described by the equation. Then, determine the type of curve described by the intersection of the surface and the given plane:

(a) **(3 points)** $1 = \frac{x^2}{12} + \frac{y^2}{17} + \frac{z^2}{2}$ and $x = 2$.
Quadric Surface: ellipsoid

Intersection with $x = 2$: ellipse

(b) **(3 points)** $1 = \frac{x^2}{12} - \frac{y^2}{17} + \frac{z^2}{2}$ and $z = 1$.
Quadric Surface: Hyperboloid of one sheet

Intersection with $z = 1$: Hyperbola

(c) **(4 points)** $0 = \frac{x^2}{12} - \frac{y^2}{17} - \frac{z}{2}$ and $y = 2$.
Quadric Surface: Hyperbolic paraboloid

Intersection with $y = 2$: parabola

14. **(10 points)** Compute the derivative of $\vec{r}(t) = \langle 2t^2 - t, \frac{4}{t}, e^{t-2} \rangle$ and determine the unit tangent vector to $\vec{r}$ at $t = 2$. What points are the initial and terminal points of this vector?

$r'(t) = \langle 4t - 1, \frac{-4}{t^2}, e^{t-2} \rangle$

$r'(2) = \langle 7, -1, 1 \rangle$

$|r'(2)| = \sqrt{49+1+1} = \sqrt{51}$

$u(2) = \langle \frac{7}{\sqrt{51}}, \frac{-1}{\sqrt{51}}, \frac{1}{\sqrt{51}} \rangle$

Initial $= r(2) = \langle 6, 2, 1 \rangle$

Terminal $= \langle 6 + \frac{7}{\sqrt{51}}, 2 - \frac{1}{\sqrt{51}}, 1 + \frac{1}{\sqrt{51}} \rangle$

15. **(10 points)** Find $\vec{r}(t)$ if $\vec{r}\,'(t) = \langle \sin(t), \cos(t), e^t \rangle$ and $\vec{r}(0) = \langle 2, 3, -4 \rangle$.

$r(t) = \int r'(t)\,dt = \langle -\cos(t), \sin(t), e^t \rangle + \langle c_x, c_y, c_z \rangle$

$r(0) = \langle 2, 3, -4 \rangle = \langle -1, 0, 1 \rangle + \langle c_x, c_y, c_z \rangle$

$r(t) = \langle 3 - \cos(t), 3 + \sin(t), e^t - 5 \rangle$

16. **(10 points)** Let $\vec{r}(t) = \langle \frac{1}{2}t^2, 17, \sqrt{2}t^2 \rangle$

(a) **(6 points)** Find the arc length function for $\vec{r}$ starting at $t = 0$.

$r' = \langle u, 0, 2\sqrt{2}u \rangle$

$|r'| = \sqrt{u^2 + 8u^2} = 3u$

$S(t) = \int_0^t |r'(u)|\,du = \int_0^t 3u\,du = \frac{3t^2}{2}$

(b) **(4 points)** Compute the length of $\vec{r}$ from $t = 2$ to $t = 4$.

$S(4) = \frac{3(4)^2}{2} = 24$

$S(2) = \frac{3(2)^2}{2} = 6$

$S(4) - S(2) = 18$

17. **(10 points)** Let $\vec{r}(t) = \langle 4t, 3\cos(t), 3\sin(t)\rangle$ and consider the point $P = (8\pi, 3, 0)$ which lies on the curve of $\vec{r}$.

$\hookrightarrow t = 2\pi$

(a) **(2 points)** Find the unit tangent vector to $\vec{r}$ at P.

$r'(t) = \langle 4, -3\sin(t), 3\cos(t)\rangle \qquad |r'(t)| = \sqrt{16+9} = 5$

$u(t) = \langle \frac{4}{5}, \frac{-3}{5}\sin(t), \frac{3}{5}\cos(t)\rangle$

$u(2\pi) = \langle \frac{4}{5}, 0, \frac{3}{5}\rangle$

(b) **(2 points)** Find the unit normal vector to $\vec{r}$ at P.

$u'(t) = \langle 0, -\frac{3}{5}\cos(t), -\frac{3}{5}\sin(t)\rangle \qquad |u'(t)| = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$N(t) = \langle 0, -\cos(t), -\sin(t)\rangle$

$N(2\pi) = \langle 0, -1, 0\rangle$

(c) **(2 points)** Find the binormal vector to $\vec{r}$ at P.

$B(t) = u(t) \times N(t) \quad \begin{vmatrix} i & j & k \\ \frac{4}{5} & \frac{-3}{5}\sin(t) & \frac{3}{5}\cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} = \langle \frac{3}{5}, \frac{4}{5}\sin(t), \frac{-4}{5}\cos(t)\rangle$

$B(2\pi) = \langle \frac{3}{5}, 0, \frac{-4}{5}\rangle$

(d) **(2 points)** Write the equation of the normal plane to $\vec{r}$ at P.

Normal vector is $u(2\pi)$

$\langle \frac{4}{5}, 0, \frac{3}{5}\rangle \cdot \langle x - 8\pi, y-3, z-0\rangle = 0$

$\frac{4}{5}x + \frac{3}{5}z = \frac{32\pi}{5}$

(e) **(2 points)** Write the equation of the osculating plane to $\vec{r}$ at P.

normal vector is $B(2\pi)$

$\langle \frac{3}{5}, 0, \frac{-4}{5}\rangle \cdot \langle x - 8\pi, y-3, z-0\rangle = 0$

$\frac{3}{5}x - \frac{4}{5}z = \frac{24\pi}{5}$

18. **(10 points)** Find the curvature of $\vec{r}(t) = \langle \frac{4}{3}t^{\frac{3}{2}}, t^2 + t, \sqrt{3}t \rangle$ for a general t and evaluate the specific point $(36, 90, 9\sqrt{3})$.

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$$r(t) = \langle \frac{4}{3} t^{\frac{3}{2}}, t^2 + t, \sqrt{3} t \rangle$$

$$r'(t) = \langle 2t^{\frac{1}{2}}, 2t+1, \sqrt{3} \rangle$$

$$r''(t) = \langle t^{-\frac{1}{2}}, 2, 0 \rangle$$

numerator:

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 2t^{\frac{1}{2}} & 2t+1 & \sqrt{3} \\ t^{-\frac{1}{2}} & 2 & 0 \end{vmatrix} = \langle -2\sqrt{3}, \sqrt{3}\, t^{-\frac{1}{2}}, 2t^{\frac{1}{2}} - t^{-\frac{1}{2}} \rangle$$

$$|r' \times r''| = \sqrt{12 + 3t^{-1} + 4t - 4 + t^{-1}} = \sqrt{4(t + 2 + t^{-1})} = 2(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$$

Denominator:

$$|r'| = \sqrt{4t + 4t^2 + 4t + 1 + 3} = \sqrt{(2t+2)^2} = 2t+2$$

$$K(t) = \frac{2(t^{\frac{1}{2}} + t^{-\frac{1}{2}})}{(2t+2)^3}$$

The point $(36, 90, 9\sqrt{3})$ occurs at $t=9$

and

$$K(9) = \frac{2(3 + \frac{1}{3})}{(20)^3} = \frac{1}{1200}$$