MEASURING DISTANCES FROM POINTS TO PLANES

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1. Introduction

In this document I want to return to an idea that we discussed in class about a month ago: computing the distance between a point $P = (x_0, y_0, z_0)$ and a plane $ax+by+cz+d = 0$. Originally, we defined this distance by picking an arbitrary point $Q = (x, y, z)$ on the plane, and projecting the vector from P to Q onto the normal vector of the plane, $\langle a, b, c\rangle$. Chasing through some algebra led us to the compact formula below:

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{1.1}$$

This is a nice application of some of our vector methods but is not the most intuitive way to approach this problem. In fact, there are several other ways to compute this distance, related to material that we have been discussing recently in class. In order to highlight the features of these other methods we will solve the specific example of finding the distance between the point $(1, 2, 3)$ and the plane $-2x - y + z - 3 = 0$ with each approach.

Using the equation in (1.1) tells us that the distance is:

$$\frac{|-2\,(1) - 1\,(2) + 1\,(3) - 3|}{\sqrt{(-2)^2 + (-1)^2 + 1^2}} = \frac{|-4|}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

The next section uses a normal line to find the closest point on the plane and then uses both the distance formula and the path length formula to compute the actual distance. Next, we will formulate the distance problem in terms of optimization and find the distance as a local minimum. Finally, the optimization problem can be described in terms of a constraint which allows us to use the method of Lagrange Multipliers to find the distance.

2. Normal Vector

Our first approach is geometrically motivated, using the fact that we know the shortest distance from the point to the plane is the length of the shortest line segment connecting $(1, 2, 3)$ to the plane. The direction vector of this line segment must be perpendicular to the plane. For our example, the direction is $\langle -2, -1, 1\rangle$ and the parametric equation of the line passing through $(1, 2, 3)$ is given by $\ell\,(t) = \langle 1 - 2t, 2 - t, 3 + t\rangle$. The intersection of this line and the plane is then the closest point on the plane to $(1, 2, 3)$. To solve for this point we can substitute the line values into the equation of the plane and solve for t:

$$\begin{aligned} -2\,(1 - 2t) - 1\,(2 - t) + 1\,(3 + t) - 3 =& 0 \\ 6t - 4 =& 0 \\ t =& \frac{2}{3} \end{aligned}$$

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At $t = \frac{2}{3}$, the line is at the point $\left(1 - 2\left(\frac{2}{3}\right), 2 - \left(\frac{2}{3}\right), 3 + \left(\frac{2}{3}\right)\right) = \left(-\frac{1}{3}, \frac{4}{3}, \frac{11}{3}\right)$, so this must be the point on the plane closest to $(1, 2, 3)$. We can check to make sure this point is on the plane by substituting in to the plane equation:

$$-2\left(-\frac{1}{3}\right) - 1\left(\frac{4}{3}\right) + 1\left(\frac{11}{3}\right) - 3 = 0$$

In order to compute the distance between the points we can either use the distance formula or compute the arc–length along the line.

2.1. **Distance Formula.** Using the distance formula we compute:

$$\sqrt{\left(1 - \frac{-1}{3}\right)^2 + \left(2 - \frac{4}{3}\right)^2 + \left(3 - \frac{11}{3}\right)^2} = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{24}{9}} = \frac{2\sqrt{6}}{3}$$

which matches up with our previous answer.

2.2. **Path Length.** Alternatively, if we want to stretch our math muscles a little more we can compute the arc length of the line $\ell(t) = \langle 1 - 2t, 2 - t, 3 + t \rangle$ from $t = 0$ to $\frac{2}{3}$

$$\begin{aligned}
\int_0^{\frac{2}{3}} |\ell'(t)| dt &= \int_0^{\frac{2}{3}} |\langle -2, -1, 1 \rangle| dt \\
&= \int_0^{\frac{2}{3}} \sqrt{(-2)^2 + (-1)^2 + 1^2} dt \\
&= \int_0^{\frac{2}{3}} \sqrt{6} dt \\
&= \sqrt{6} \int_0^{\frac{2}{3}} dt \\
&= \frac{2\sqrt{6}}{3}
\end{aligned}$$

which is starting to look fairly familiar at this point.

3. Distance Minimization

We can also formulate this problem in terms of optimization. We begin by noting that the distance from any point (x, y, z) on the plane to $(1, 2, 3)$ is given by $d(x, y, z) = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$. Thus, if we can find the point that minimizes this function we will have found the distance. The squareroot is a little unpleasant to deal with, particularly if we are going to be taking derivatives, so instead we are going to focus on optimizing the square of the distance function, $F(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$ since they are both minimized at the same point. In order to make use of the techniques that we have used in class, we also need to convert this into a function of 2 variables.

We can do this using the constraint the point (x, y, z) must lie on the plane. This implies that $-2x - y + z - 3 = 0$ or $z = 3 + 2x + y$. Making this substitution into F gives:

$$\begin{aligned}
f(x, y) &= F(x, y, 3 + 2x + y) \\
&= (x-1)^2 + (y-2)^2 + ((3 + 2x + y) - 3)^2 \\
&= x^2 - 2x + 1 + y^2 - 4y + 4 + 4x^2 + 4xy + y^2 \\
&= 5x^2 - 2x + 4xy + 2y^2 - 4y + 5
\end{aligned}$$

which is now a function that we know how to optimize. We first look for the critical points by computing the partial derivatives and setting them equal to zero:

$$\begin{aligned} f_x =& 10x - 2 + 4y = 0 \\ f_y =& 4\left(y + x - 1\right) = 0 \end{aligned}$$

The second equation tells us that $x + y = 1$ or $y = 1 - x$. Substituting this into the first equation gives:

$$\begin{aligned} 10x + 4\left(1 - x\right) =& 2 \\ 6x =& -2 \\ x =& -\frac{1}{3} \end{aligned}$$

Now that we have the x value we can get y and z by substitution as well:

$$\begin{aligned} y =& 1 - x = \frac{4}{3} \\ z =& 3 + 2x + y = \frac{11}{3} \end{aligned}$$

Again, we have discovered that the closest point to $(1, 2, 3)$ on the plane is $\left(-\frac{1}{3}, \frac{4}{3}, \frac{11}{3}\right)$. To check that this point minimizes the distance function we apply the second derivative test:

$$\begin{aligned} f_{xx} =& 10 \\ f_{yy} =& 4 \\ f_{xy} =& 4 \end{aligned}$$

with $D = f_{xx}f_{yy} - f_{xy}^2 = 40 - 16 = 24 > 0$ and $f_x x = 10 > 0$ so $\left(-\frac{1}{3}, \frac{4}{3}, \frac{11}{3}\right)$ is a local minimum and the value of the function at the point is:

$$f\left(-\frac{1}{3}, \frac{4}{3}, \frac{11}{3}\right) = \left(-\frac{1}{3} - 1\right)^2 + \left(\frac{4}{3} - 2\right)^2 + \left(\frac{11}{3} - 3\right)^2 = \frac{8}{3} = \left(\frac{2\sqrt{6}}{3}\right)^2$$

exactly the square of the distance.

4. Lagrange Multipliers

Finally, we can solve this problem using the idea of Lagrange Multipliers. We will again try to minimize the square of the distance formula, $F\left(x, y, z\right) = \left(x - 1\right)^2 + \left(y - 2\right)^2 + \left(z - 3\right)^2$, but instead of rewriting this as a two variable optimization problem we will encode the constraint (that the point (x, y, z) lie on the plane) as a function $g\left(x, y, z\right) = -2x - y + z - 3 = 0$. To solve this problem we need to find all values of (x, y, z) that satisfy: $\nabla\left(f\right)\left(x, y, z\right) = \lambda\nabla\left(g\right)\left(x, y, z\right)$ and $g\left(x, y, z\right) = 0$. This gives us four equations in four unknowns:

$$\begin{aligned} f_x = 2x - 2 =& -2\lambda = \lambda g_x \\ f_y = 2y - 4 =& -1\lambda = \lambda g_y \\ f_z = 2z - 6 =& -1\lambda = \lambda g_z \\ g\left(x, y, z\right) =& -2x - y + z - 3 = 0 \end{aligned}$$

We can solve the first three equations for x, y, and z in terms of λ:

$$x = 1 - \lambda$$
$$y = 2 - \frac{\lambda}{2}$$
$$z = 3 + \frac{\lambda}{2}$$

Substituting these into the fourth equation lets us solve for λ directly:

$$-2\left(1-\lambda\right) - 1\left(2 - \frac{\lambda}{2}\right) + \left(3 + \frac{\lambda}{2}\right) - 3 = 0$$
$$3\lambda - 4 = 0$$
$$\lambda = \frac{4}{3}$$

Plugging this value of λ in for our previous equations gives the very familiar result:

$$x = -\frac{1}{3}$$
$$y = \frac{4}{3}$$
$$z = \frac{11}{3}$$

Again, we have found that the closest point is $\left(-\frac{1}{3}, \frac{4}{3}, \frac{11}{3}\right)$ and the value of the function at the point is:

$$f\left(-\frac{1}{3}, \frac{4}{3}, \frac{11}{3}\right) = \left(-\frac{1}{3} - 1\right)^2 + \left(\frac{4}{3} - 2\right)^2 + \left(\frac{11}{3} - 3\right)^2 = \frac{8}{3} = \left(\frac{2\sqrt{6}}{3}\right)^2$$

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