TAYLOR POLYNOMIALS

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1. Introduction

Taylor polynomials are a generalization of the linearizations that we studied earlier in the week. The linearization of a function f at a point a is the function given by

$$L(x) = f(a) + f'(a)(x - a).$$

This is the same as the equation of the tangent line to f at a, and we have seen that for values of x close to a that $f(x) \approx L(x)$. The linearization uses two pieces of information, the value of f at a and the instantaneous rate of change of f at a, to provide a simple approximation. If we want to include more information about f in an approximation we need a higher degree polynomial.

2. Taylor Polynomials

The Taylor polynomial of degree n, denoted $T_n(x)$, allows us to approximate f near a point a with a degree n polynomial whose first n derivatives at a are the same as the first n derivatives of f at a. The linearization discussed above is the first Taylor polynomial: $T_1(x)$ – its derivative at a is exactly the same as the derivative of f at a by construction. To construct this polynomial we need to know the values of the first n derivatives of f at a. Then, the Taylor polynomial can be written as:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Sometimes, in order to simplify the notation, we write this as:

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$$

with

$$c_0 = f(a)$$

and

$$c_k = \frac{f^{(k)}(a)}{k!}.$$

3. four step process

This leads us to the following four–step process for finding Taylor polynomials:

(1) Compute the first n derivatives of f
(2) Compute $f(a), f'(a), f''(a), \ldots, f^{(n)}(a)$
(3) Compute $c_0, c_1, \ldots, c_n$
(4) Substitute in to the formula for $T_n(x)$

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4. Examples

Example 1. *Find the fourth degree Taylor polynomial of* $f(x) = e^{2x} + x^2$ *near* $a = 0$.

(1) *First, we need to compute the first* 4 *derivatives of* f*:*

$$\begin{aligned} f(x) &= e^{2x} - x^2 \\ f'(x) &= 2e^{2x} - 2x \\ f''(x) &= 4e^{2x} - 2 \\ f'''(x) &= 8e^{2x} \\ f''''(x) &= 16e^{2x} \end{aligned}$$

(2) *Next, we need to compute the values of the derivatives at* $a = 0$*:*

$$\begin{aligned} f(0) &= e^0 - 0 = 1 \\ f'(0) &= 2e^0 - 0 = 2 \\ f''(0) &= 4e^0 - 2 = 2 \\ f'''(0) &= 8e^0 = 8 \\ f''''(0) &= 16e^0 = 16 \end{aligned}$$

(3) *Now we can solve for the coefficients:*

$$\begin{aligned} c_0 &= f(0) = 1 \\ c_1 &= \tfrac{f'(0)}{1!} = \frac{2}{1} = 2 \\ c_2 &= \tfrac{f''(0)}{2!} = \frac{2}{2} = 1 \\ c_3 &= \tfrac{f'''(0)}{3!} = \frac{8}{6} = \frac{4}{3} \\ c_4 &= \tfrac{f''''(0)}{4!} = \frac{16}{24} = \frac{2}{3} \end{aligned}$$

(4) *Finally, we can substitute:*

$$T_4(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4$$

$$T_4(x) = 1 + 2(x-0) + 1(x-0)^2 + \frac{4}{3}(x-0)^3 - \frac{2}{3}(x-0)^4$$

$$T_4(x) = 1 + 2x + x^2 + \frac{4}{3}x^3 - \frac{2}{3}x^4$$

Example 2. *Find the sixth degree Taylor polynomial of $f(x) = \sin(x) + \cos(x)$ at $a = \pi$.*

(1) First, we need to compute the first 6 derivatives of f:

$$\begin{aligned}
f(x) &= \sin(x) + \cos(x) \\
f'(x) &= \cos(x) - \sin(x) \\
f''(x) &= -\sin(x) - \cos(x) \\
f'''(x) &= -\cos(x) + \sin(x) \\
f''''(x) &= \sin(x) + \cos(x) \\
f'''''(x) &= \cos(x) - \sin(x) \\
f''''''(x) &= -\sin(x) - \cos(x)
\end{aligned}$$

(2) Next, we need to compute the values of the derivatives at $a = \pi$:

$$\begin{aligned}
f(\pi) &= \sin(\pi) + \cos(\pi) = -1 \\
f'(\pi) &= \cos(\pi) - \sin(\pi) = -1 \\
f''(\pi) &= -\sin(\pi) - \cos(\pi) = 1 \\
f'''(\pi) &= -\cos(\pi) + \sin(\pi) = 1 \\
f''''(\pi) &= \sin(\pi) + \cos(\pi) = -1 \\
f'''''(\pi) &= \cos(\pi) - \sin(\pi) = -1 \\
f''''''(\pi) &= -\sin(\pi) - \cos(\pi) = 1
\end{aligned}$$

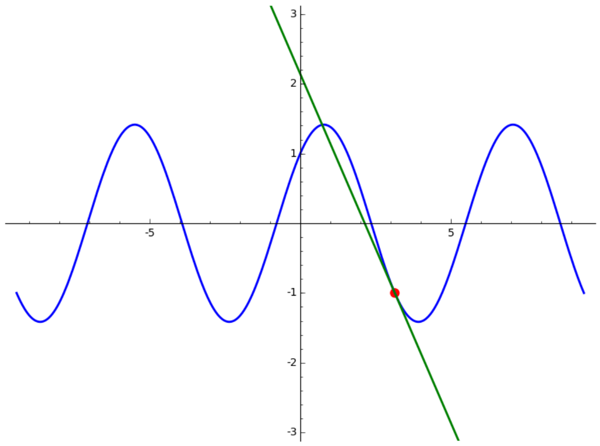
(3) Now we can solve for the coefficients:

$$\begin{aligned}
c_0 &= f(\pi) = -1 \\
c_1 &= \tfrac{f'(\pi)}{1!} = \frac{-1}{1} \\
c_2 &= \tfrac{f''(\pi)}{2!} = \frac{1}{2} \\
c_3 &= \tfrac{f'''(\pi)}{3!} = \frac{1}{6} \\
c_4 &= \tfrac{f''''(\pi)}{4!} = \frac{-1}{24} \\
c_5 &= \tfrac{f'''''(\pi)}{5!} = \frac{-1}{120} \\
c_6 &= \tfrac{f''''''(\pi)}{6!} = \frac{1}{720}
\end{aligned}$$

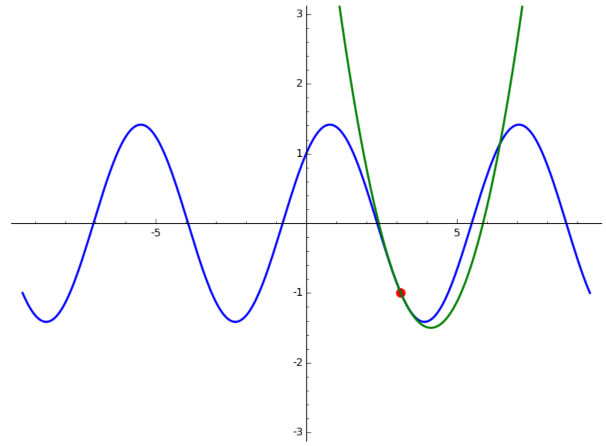
Finally, we can substitute:

$$T_6(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + c_6(x-a)^6$$

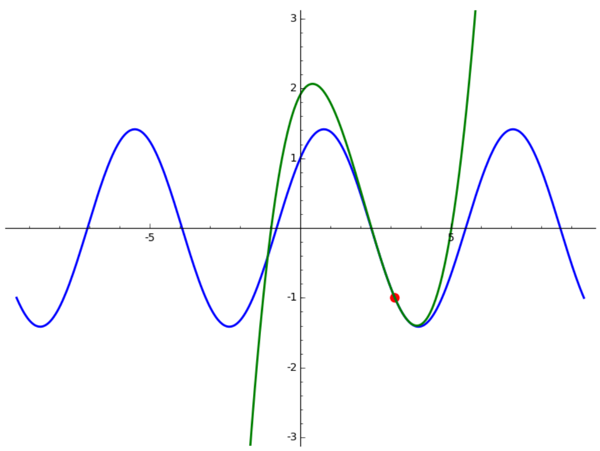
$$T_6(x) = -1 - 1(x-\pi) + \frac{1}{2}(x-\pi)^2 + \frac{1}{6}(x-\pi)^3 - \frac{1}{24}(x-\pi)^4 - \frac{1}{120}(x-\pi)^5 + \frac{1}{720}(x-\pi)^6.$$



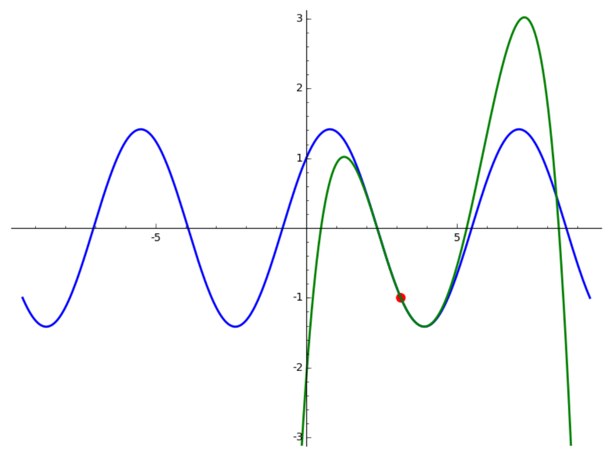
(A) $T_1(x)$



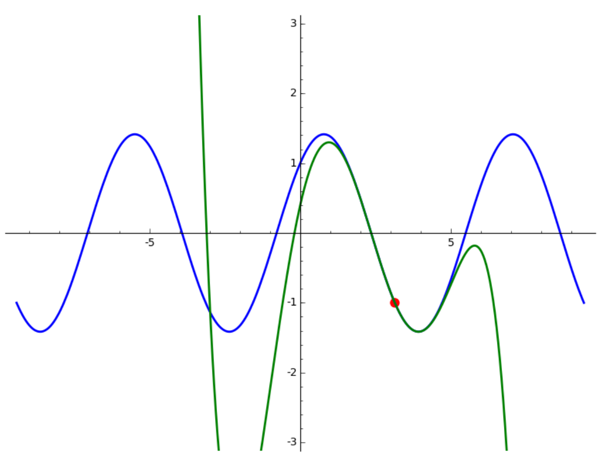
(B) $T_2(x)$



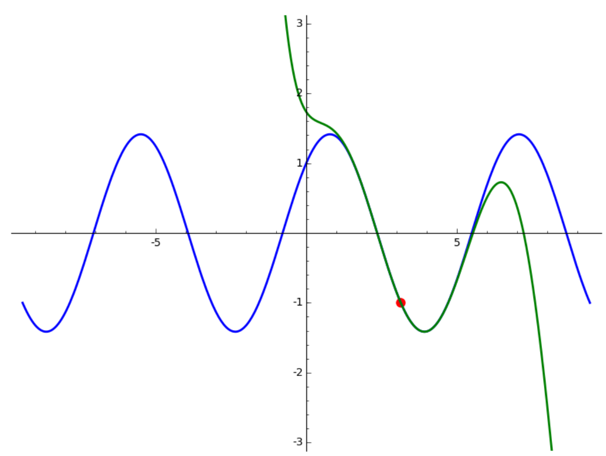
(C) $T_3(x)$



(D) $T_4(x)$



(E) $T_5(x)$



(F) $T_6(x)$

FIGURE 1. The first six Taylor Polynomials approximating $f(x) = \sin(x) + \cos(x)$ at $a = \pi$

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