Math 8- winter 2020

Answer key for written homework due 3/2.

1. Homework assigned 2/24.

Let $f(x,y) = z = x^2+y^2$.

At point (a,b), the tangent plane to $z=x^2+y^2$ is given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$= (a^2+b^2) + 2a(x-a) + 2b(y-b)$$ because $f_x(a,b) = 2a$ and $f_y(a,b) = 2b$

$$= 2ax + 2by - a^2 - b^2.$$

So this plane can be rewritten as

$$-2ax - 2by + z = -a^2 - b^2.$$

It is parallel to $2x-6y+z=5$ whenever $-2a=2$ and $-6=-2b$, that is at $(-1,3)$.

Hence, the point on the paraboloid is $(-1,3,10)$, and the tangent plane is

$$z = -2x+6y-10 \quad \text{or} \quad 2x-6y+z = -10.$$

2. Homework assigned 2/25 or 2/27.

The plane tangent to $f(x,y)$ at $(2,1,3)$ is

$$z = 4x-2y-3.$$

This is because the partial derivatives are

$$f_x(a,b) = 2a \quad \text{and} \quad f_y(a,b) = -2b,$$

which are, at $(2,1)$

$$f_x(2,1) = 4 \quad \text{and} \quad f_y(2,1) = -2.$$

Hence,

$$h(x,y) = z = 3 + 4(x-2) - 2(y-1) = 4x-2y-3.$$

The plane $z = h(x,y)$ is tangent to $f(x,y)$ if and only if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-h(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

This is equivalent to

$$\lim_{(x,y)\to(a,b)} \left| \frac{f(x,y)-h(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} \right| = 0$$

Here, that is

$$\lim_{(x,y)\to(2,1)} \left| \frac{x^2-y^2-4x+2y+3}{\sqrt{(x-2)^2+(y-1)^2}} \right| = \lim_{(x,y)\to(2,1)} \left| \frac{(x^2-4x+4)-(y^2-2y+1)}{\sqrt{(x-2)^2+(y-1)^2}} \right|$$

$$= \lim_{(x,y)\to(2,1)} \left| \frac{(x-2)^2-(y-1)^2}{\sqrt{(x-2)^2+(y-1)^2}} \right|$$

because $|a^2-b^2| < |a^2+b^2|$ and we can remove the absolute value since the result is always nonnegative

$$\le \lim_{(x,y)\to(2,1)} \frac{(x-2)^2+(y-1)^2}{\sqrt{(x-2)^2+(y-1)^2}}$$

$$= \lim_{(x,y)\to(2,1)} \sqrt{(x-2)^2+(y-1)^2} = 0$$

We just proved, using the definition of the limit, that $h(x,y)=z$ is really tangent to $f(x,y)$.

3. Homework assigned 2/27

Using the partial derivatives $g_x = 2x-3$ and $g_y = 2y$, we find the gradient $\nabla g(x,y) = \langle 2x-3, 2y\rangle$ and $\nabla g(1,2) = \langle -1, 4\rangle$. The gradient is always orthogonal to the level curve, and a vector that is orthogonal to it is $\langle 4,1\rangle$, since $\langle -1,4\rangle \cdot \langle 4,1\rangle = 0$. A point on that tangent line is $(1,2)$, so the tangent line is $\vec{L}(t) = \langle 4,1\rangle t + \langle 1,2\rangle$.

As for the level curve $g(x,y)=0$, that is

$$x^2+y^2-3x=2 \iff \left(x^2-3x+\frac{9}{4}\right)+y^2=\frac{17}{4}$$

$$\iff \left(x-\frac{3}{2}\right)^2+y^2=\left(\frac{\sqrt{17}}{2}\right)^2.$$

It is a circle of radius $\frac{\sqrt{17}}{2}$ (a little bit over 2), centered at $\left(\frac{3}{2},0\right)$. It is passing through $(1,2)$.