Math 8 - Winter 2020

Answer key for homework due 3/6

1. Homework assigned 3/2

#1. $f(x,y) = y^3 + 6x^2 + 3y^2 - 3x^2y$.

The critical points are when $f_x(x,y) = f_y(x,y) = 0$.

Here,

$$f_x(x,y) = 12x - 6xy$$

$$f_y(x,y) = 3y^2 + 6y - 3x^2.$$

To get $f_x(x,y) = 0$, we need either $x=0$ or $y=2$.

If $x=0$, $f_y = 0$ iff $y=0$ or $y=-2$.

If $y=2$, $f_y = 0$ iff $x^2 = 8 \Rightarrow x = \pm\sqrt{8}$.

The critical points are $(0,0)$, $(0,-2)$, $(\sqrt{8},2)$ and $(-\sqrt{8},2)$.

To classify them, we use the second derivative test, and need the discriminant:

$$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2.$$

Here,

$$f_{xx}(x,y) = 12 - 6y \qquad f_{xy}(x,y) = -6x$$

$$f_{yy}(x,y) = 6y + 6$$

and thus

$$D(x,y) = 36\left[-x^2 - y^2 + y + 2\right].$$

Hence,

(x,y)	$D(x,y)$	f_{xx}	min/max/saddle pt.
$(0,0)$	72	12	min
$(0,-2)$	-144		saddle pt
$(\sqrt{8},2)$	-288		"
$(-\sqrt{8},2)$	-288		"

#2 (a) $f(x,y) = x^4 + y^4$.

Here,

$f_x(x,y) = 4x^3$ $\quad$ $f_y = 4y^3$

$f_{xx}(x,y) = 12x^2$ $\quad$ $f_{yy} = 12y^2$

$f_{xy}(x,y) = 0$

Hence, the discriminant at the origin is

$$D(0,0) = \left(12x^2 \cdot 12y^2\right)\Big|_{(x,y)=(0,0)}$$

$$= 0,$$

and we cannot apply the second derivative test.

However, f is a nonnegative function and $f(0,0) = 0$, so the origin is a minimum.

(b) $f(x,y) = x^4 - x^2y^2$

Here,

$f_x(x,y) = 4x^3 - 2xy^2$ $\quad$ $f_y(x,y) = -2x^2y$

$f_{xx}(x,y) = 12x^2 - 2y^2$ $\quad$ $f_{yy}(x,y) = -2x^2$

$f_{xy}(x,y) = -4xy$

Hence, the discriminant is

$$D(x,y) = (12x^2 - 2y^2)(-2x^2) - (-4xy)^2$$

and $D(0,0) = 0$, so the test fails.

However, we can say it is a saddle point by comparing the intersection of the graph of f with two planes:

- with $y = 0$; and we get the function $g(x) = x^4$, which admits a minimum at 0.
- with $y = 2x$; and we get $h(x) = -3x^4$, which admits a maximum at 0.

Since it is a maximum in one direction and a minimum in another, it is a saddle point.

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2. Homework assigned 3/4.

Let $f(x,y) = xy$.

To find all the extreme values on $4x^2+y^2 \le 4$, we need to

(i) find the critical points of f within $4x^2+y^2 < 4$, and evaluate f at these points.

(ii) apply the method of Lagrange multipliers with the constraint $4x^2+y^2=4$.

(iii) Compare answers from both parts.

(i) The critical points occur when $f_x(x,y) = f_y(x,y) = 0$:

$$f_x = y \quad \text{and} \quad f_y = x.$$

Hence, the only critical point is $(0,0)$, and $f(0,0) = 0$.

(ii) Let $g(x,y) = 4x^2+y^2 = 4$.

$$\nabla g(x,y) = \langle 8x, 2y \rangle.$$

$\nabla g(x,y)$ is parallel to $\nabla f(x,y) = \langle y, x \rangle$ when $\lambda \nabla g(x,y) = \nabla f(x,y)$, that is when

$$8\lambda x = y \quad \text{and} \quad 2\lambda y = x.$$

Solving for λ, we get $\lambda = \pm\frac{1}{4}$ (or $(x,y) = (0,0)$, but that does not satisfy the constraint).

That means that $y^2 = 4x^2$. Since $4x^2+y^2=4$, that is $y = \pm\sqrt{2}$ and $x = \frac{\pm 1}{\sqrt{2}}$.

Hence, extreme values may occur at $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(\frac{-1}{\sqrt{2}}, \sqrt{2}\right), \left(\frac{1}{\sqrt{2}}, -\sqrt{2}\right), \left(\frac{-1}{\sqrt{2}}, -\sqrt{2}\right)$ and $(0,0)$:

$f\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right) = 1 = f\left(\frac{-1}{\sqrt{2}}, -\sqrt{2}\right)$, $f\left(\frac{-1}{\sqrt{2}}, \sqrt{2}\right) = f\left(\frac{1}{\sqrt{2}}, -\sqrt{2}\right) = -1$, $f(0,0) = 0$.

Hence, the maximum value is 1, reached at $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(\frac{-1}{\sqrt{2}}, -\sqrt{2}\right)$, and the minimum value is -1, reached at $\left(\frac{-1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$.