Math 8
Winter 2020

WeBWorK Day 5 Problem 3
Sample Solution

Note: This problem is slightly more complex than the WeBWorK problem assigned for homework. This is because many students did not have the correct method for finding the radius of convergence, even though they arrived at the correct answer. This solution will illustrate a correct way to find the radius of convergence.

Problem: The function $f(x) = 3x^2 \arctan(8x^3)$ is represented as a power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

What is the lowest term with a nonzero coefficient ($n =$ _____)? What is the radius of convergence ($R =$ _____)?

Solution: Taylor series are examples of power series. So if we know the Taylor series for $f(x)$ converges to $f(x)$, then that Taylor series *is* the power series that represents $f(x)$.

We know (from the notes or textbook) a Taylor series that converges to $\arctan(x)$ (a power series that represents $\arctan(x)$), so we start with that:

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad \text{for } |x| \leq 1.$$

We can get a power series that represents $\arctan(8x^3)$ by plugging in $8x^3$ in place of x:

$$\arctan(8x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k (8x^3)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 8^{2k+1} x^{6k+3}}{2k+1} =$$

$$8x^3 - \frac{8^3 x^9}{3} + \frac{8^5 x^{15}}{5} - \cdots \qquad \text{for } |8x^3| \leq 1.$$

Our function is $3x^2$ times $\arctan(8x^3)$, so we can multiply our equation by $3x^2$ on both sides:

$$3x^2 \arctan(8x^3) = 3x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 8^{2k+1} x^{6k+3}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k (3) 8^{2k+1} x^{6k+5}}{2k+1} =$$

$$3 \cdot 8x^5 - \frac{3 \cdot 8^3 x^{11}}{3} + \frac{3 \cdot 8^5 x^{17}}{5} - \cdots \qquad \text{for } |8x^3| \leq 1.$$

We have found

$$f(x) = 3x^2 \arctan(8x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k (3) 8^{2k+1} x^{6k+5}}{2k+1} =$$

$$3 \cdot 8x^5 - \frac{3 \cdot 8^3 x^{11}}{3} + \frac{3 \cdot 8^5 x^{17}}{5} - \cdots \qquad \text{for } |8x^3| \leq 1.$$

Question 1: What is the radius of convergence ($R =$ _____)? For the arctangent function we had radius of convergence $R = 1$, or convergence for $|x| \leq 1$. When we made our substitution of $8x^3$ in place of x, we got convergence for $|8x^3| \leq 1$, which we can rewrite as $|x^3| \leq \frac{1}{8}$, or $|x| \leq \frac{1}{2}$. This means our radius of convergence is $\boxed{R = \frac{1}{2}}$.

Question 2: What is the lowest term ($n =$ _____) with a nonzero coefficient? To answer this question, we need to look at the way the original problem wrote its power series representation,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

The question asks for the first n for which $c_n \neq 0$.

Your expression is the power series representation for $f(x)$, even if it doesn't look the same.

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} \frac{(-1)^k (3) 8^{2k+1} x^{6k+5}}{2k+1}.$$

Writing out the power series term-by-term, we have

$$c_0 + c_1 x + c_2 x^2 + \cdots = 3 \cdot 8x^5 - \frac{3 \cdot 8^3 x^{11}}{3} + \frac{3 \cdot 8^5 x^{17}}{5} - \cdots.$$

Because the right hand side doesn't have a constant term, the constant c_0 must equal zero. In the same way, the right hand side doesn't have an x term, so $c_1 = 0$. The first nonzero c_n appears in the first place the right hand side has a nonzero multiple of x^n. That is the x^5 term, so the answer is $\boxed{n = 5}$.