Math 8
Winter 2020

WeBWorK Day 7 Problem 5
Sample Solution

Problem: Let $F(a)$ be the area between the x-axis and the graph of $y = x^2 e^{-\frac{x}{3}}$ between $x = 0$ and $x = a$ for $a > 0$ (consider the area to be negative if the graph lies below the x-axis).

Question 1: Find a formula for $F(a)$.
Solution: Start with

$$F(a) = \int_0^a x^2 e^{-\frac{x}{3}}\,dx.$$

We use integration by parts twice. In the first integral, we use $u = x^2$ and $dv = e^{-\frac{x}{3}}\,dx$, so $du = 2x\,dx$ and $v = -3e^{-\frac{x}{3}}$.

$$\int x^2 e^{-\frac{x}{3}}\,dx = \int u\,dv = uv - \int v\,du = -3x^2 e^{-\frac{x}{3}} - \int (-3e^{-\frac{x}{3}})2x\,dx = -3x^2 e^{-\frac{x}{3}} + 6\int x e^{-\frac{x}{3}}\,dx.$$

In the second integral, we use $u = x$ and $dv = e^{-\frac{x}{3}}\,dx$, so $du = dx$ and $v = -3e^{-\frac{x}{3}}$.

$$\int x e^{-\frac{x}{3}}\,dx = \int u\,dv = uv - \int v\,du = -3xe^{-\frac{x}{3}} - \int (-3e^{-\frac{x}{3}})\,dx = -3xe^{-\frac{x}{3}} + 3\int e^{-\frac{x}{3}}\,dx$$
$$= -3xe^{-\frac{x}{3}} - 9e^{-\frac{x}{3}} + C.$$

We substitute back into our first integral:

$$\int x^2 e^{-\frac{x}{3}}\,dx = -3x^2 e^{-\frac{x}{3}} + 6\int x e^{-\frac{x}{3}}\,dx = -3x^2 e^{-\frac{x}{3}} + 6(-3xe^{-\frac{x}{3}} - 9e^{-\frac{x}{3}}) + C =$$
$$-3x^2 e^{-\frac{x}{3}} - 18xe^{-\frac{x}{3}} - 54e^{-\frac{x}{3}} + C.$$

Now we can evaluate $F(a)$:

$$F(a) = \int_0^a x^2 e^{-\frac{x}{3}}\,dx = \left(-3x^2 e^{-\frac{x}{3}} - 18xe^{-\frac{x}{3}} - 54e^{-\frac{x}{3}}\right)\Big|_{x=0}^{a} = -3a^2 e^{-\frac{a}{3}} - 18ae^{-\frac{a}{3}} - 54e^{-\frac{a}{3}} + 54.$$

Question 2: Is $F(a)$ increasing or decreasing for $0 < a < 2$?
Solution: We can find $F'(a)$ by differentiation, or by using the Fundamental Theorem of Calculus:

$$F'(a) = a^2 e^{-\frac{a}{3}}.$$

Since this is always positive, $F(a)$ is increasing.
Question 3: Is $F(a)$ concave up or concave down for $0 < a < 2$?
Solution: We find $F''(a)$ by differentiation,

$$F''(a) = \left(2a - \frac{a^2}{3}\right)e^{-\frac{a}{3}} = a\left(2 - \frac{a}{3}\right)e^{-\frac{a}{3}}.$$

Since this is positive for $0 \le a \le 2$, on this interval $F(a)$ is concave upward.