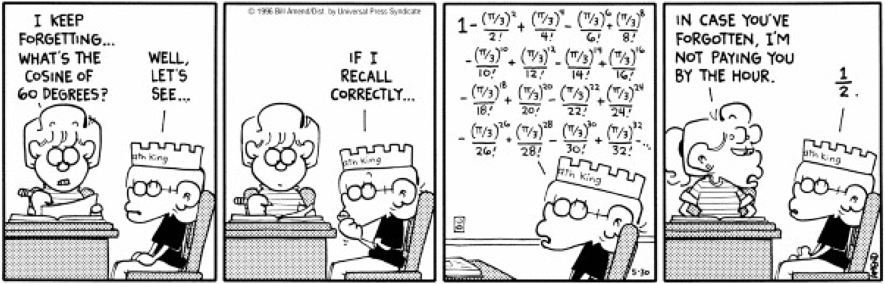
Taylor Polynomials



Goal and motivation

We want to approximate certain functions with polynomials (such that they have the same derivatives).

Such approximations are also useful to

- integrate functions that may not have an elementary antiderivative (e.g. $f(x) = e^{x^2}$)
- to solve some differential equations (i.e. equations involving derivatives)

We already know such an approximation... **linear aproximation**:

Given a differentiable function $f : \mathbb{R} \to \mathbb{R}$, then the linear approximation of f at $x = a$ is given by

$$T(x) = f(a) + f'(a)(x - a)$$

Idea: approximate a function with a straight line close to a given point (i.e. use the tangent to its graph).

Example: The linear approximation of $f(x) = e^x$ at $x = 0$ is given by

$$T(x) = e^0 + e^0(x - 0) = 1 + x$$

Unfortunately, this is a bad approximation further away from 0. E.g. $f(4) = 54.59815003$, but $T(4) = 5$.

An improved approximation

Let's explore $T(x) = f(a) + f'(a)(x - a)$, for some function f at $x = a$, and build on its properties to find better approximations.
We note that

- $T(x) = f'(a)x + constant$, that is, $T(x)$ is a 1^{st}-degree polynomial
- $T(a) = f(a)$ and $T'(a) = f'(a)$, i.e. they have the same derivatives at $x = a$.

New idea: approximate f by a polynomial T_n (of arbitrary degree n) such that derivatives of all orders are equal (i.e. $T_n^{(k)}(a) = f^{(k)}(a)$).
Hopefully the approximation gets better as we increase the degree of the polynomial...

Taylor Polynomials

Suppose f has continuous derivatives of all orders.

Definition

The n^{th}**-degree Taylor polynomial** of f at $x = a$ is

$$\begin{aligned} T_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{2\cdot 3}(x-a)^3 + \\ &\quad \ldots + \frac{f^{(n)}(a)}{2\cdot 3\cdots n}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

When $a = 0$ we call T_n a **Maclaurin polynomial**.

▸ Tyalor polynimial approximation

Examples

Find

1. the 4^{th}-degree Taylor polynomial for $f(x) = e^x$ at $x = 0$,
2. the 4^{th}- and 40^{th}-degree Taylor polynomials for $g(x) = \cos(x)$ at $x = 0$,
3. the 20^{th}-degree Taylor polynomial for $h(x) = x^{-2}$ at $x = 1$.
4. the 70^{th}-degree Maclaurin polynomial for $y(x) = \sin(3x)$ at $x = \pi$.

How good are these approximations?
Consider ▸ $h(x) = x^{-2}$ at $x = 1$

Error

Definition

The n^{th} **remainder** is defined by

$$R_n(x) = f(x) - T_n(x).$$

The **error** in the approximation is

$$\mid R(x) \mid .$$

We hope that

$$\lim_{n\to\infty} R(x) = 0$$

or equivalently, that

$$\lim_{n\to\infty} T_n(x) = f(x).$$

Examples

We'll have more to say about the following important example that will be used later on again.

$$f(x) = \frac{1}{1-x}$$

(1) Show the Maclaurin polynomial for f is $T_n(x) = \dfrac{1-x^{1+n}}{1-x}$

(2) Show that $T_n(x) = f(x) - \dfrac{x^{1+n}}{1-x}$. So $R_n(x) = \dfrac{x^{1+n}}{1-x}$

(3) Compute $\lim_{n\to\infty} R_n(3)$ and $\lim_{n\to\infty} R_n(\frac{1}{3})$. What can you say about T_n as an approximation for f?