Vectors

KeyConcept **Dot Product of Vectors in a Plane**

The dot product of $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$.

Does this remind you of anything? What? How?

$$\begin{bmatrix} r_1 & r_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = [\, r_1 d_1$$

$]$

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

Definition

The **dot product** of $\vec{a}$ and $\vec{b}$ is the *scalar* defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Properties of the dot product

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be coordinate vectors in $\mathbb{R}^n$ and let $k \in \mathbb{R}$ be a scalar. Then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b})$
5. $\mathbf{a} \cdot \mathbf{0} = \mathbf{0}$

Angles between vectors

Theorem

If $0 \leq \theta \leq \pi$ is the angle between $\vec{a}$ and $\vec{b}$, then

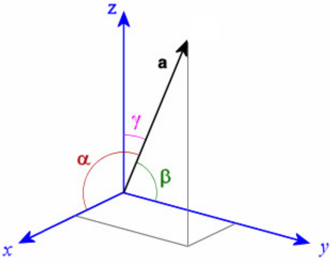
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta) \;\Rightarrow\; \cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Theorem

$$\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$$

- The dot product is defined for vectors, but it returns a *scalar*
- The dot product is useful for finding the angle between two vectors. In particular, to see of two vectors are perpendicular (or orthogonal).
- The dot product is useful for finding the components of vectors in certain directions - i.e. projections

Angles with the axes

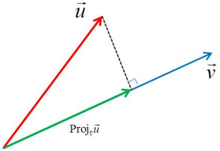


The **direction angles** of non-zero $\vec{a}$ the the angles α, β and γ that $\vec{a}$ makes with the positive $x-, y-$ and $z-$axes, respectively.
The **direction cosines** are the cosines of these angles.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\cos(\alpha) = \frac{a_1}{|\vec{a}|}, \ \cos(\beta) = \frac{a_2}{|\vec{a}|}, \ \cos(\gamma) = \frac{a_3}{|\vec{a}|}$$

Projections



Scalar projection

The scalar projection of $\vec{u}$ onto $\vec{v}$ is

$$|\vec{u}|\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

NB: This is a scalar, i.e. the length of $\vec{u}$'s projection onto $\vec{v}$

Vector projection

The vector projection of $\vec{u}$ onto $\vec{v}$ is

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$$