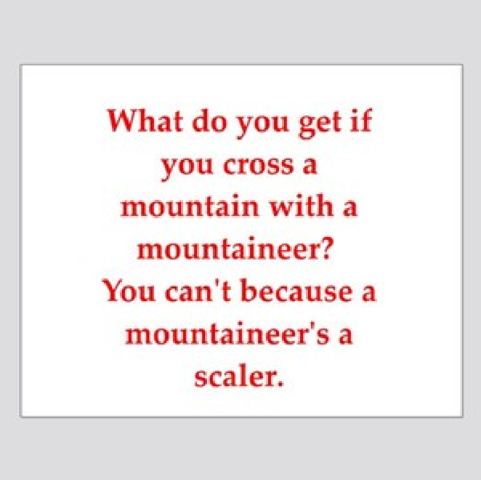
Cross Product



Definition

The **cross product** of two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is defined as

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

We don't need to remember this formula - only how to compute determinants.

Example: Compute $\vec{a} \times \vec{b}$ if $\vec{a} = \langle 1, 1, 0 \rangle$ and $\vec{b} = \langle 2, 0, -2 \rangle$

Theorem

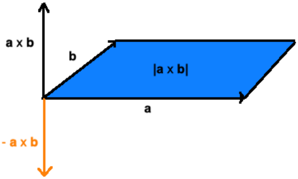
If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\theta)$$

Corollary: $\vec{a} || \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$

- The cross product is applied to vectors and returns a **vector**.
- $(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp \vec{b}$.
- The direction of $\vec{a} \times \vec{b}$ can be determined with the right-hand rule.
- The cross product is useful to find a vector orthogonal to two given vectors (or a given plane, for example).
- The cross product is also useful to compute certain areas and volumes.

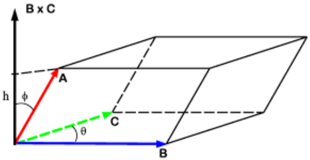
Area of a Parallelogram



Area of parallelogram formed by $\vec{a}$ and $\vec{b}$ is

$$A = |\vec{a} \times \vec{b}|$$

Volume of a Parallelepiped



Volume of a parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

Properties of the cross product

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be coordinate vectors in $\mathbb{R}^n$ and let $k \in \mathbb{R}$ be a scalar. Then

1. $\mathbf{a} \perp \mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \perp \mathbf{a} \times \mathbf{b}$
2. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
3. $(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (k\mathbf{b})$
4. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
5. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
6. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ (scalar triple product)
7. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$