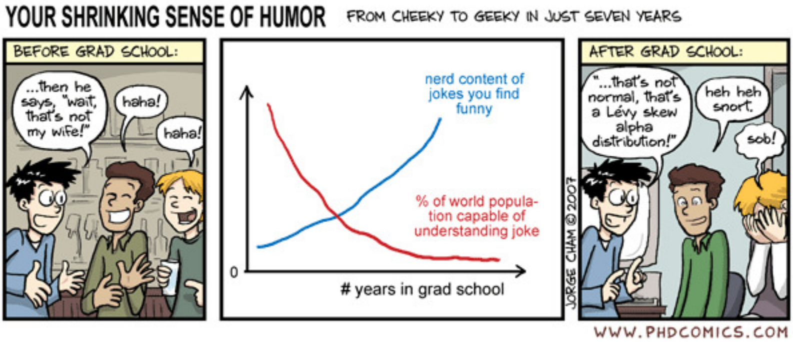
Differentiation and Integration of Vector Functions



Limits and continuity

Limit

If $\vec{r}(t) =< f(t), g(t), h(t) >$, then

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle,$$

provided **each** component-limit exists.

Continuity

$\vec{r}(t) =< f(t), g(t), h(t) >$ is continuous at $t = a$ if and only if

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$$

(i.e. if and only if **each component function is continuous**)

Derivatives

Derivative of $\vec{r}$

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

More user-friendly is:

Theorem

If $\vec{r}(t) =< f(t), g(t), h(t) >$ *and* f', g' *and* h' *all exist, then*

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

Theorem

If $\vec{r}(t) =< f(t), g(t), h(t) >$ *and* f', g' *and* h' *all exist, then*

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle.$$

The **unit tangent vector** is given by

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Example: Find the equation of the tangent line to $\vec{r}(t) = \langle \sin(2t), e^{5t}, 3t\rangle$ at $t = 0$.

Smoothness

A curve $\vec{r}(t), t \in I$, is **smooth** if $\vec{r}\,'$ is <u>continuous</u> and $\underline{\vec{r}\,'(t) \neq \vec{0}}$ for every $t \in I$ (except maybe endpoint of I).

Differentiation Rules

3 THEOREM Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Integrals

Definite integral of $\vec{r}(t) =< f(t), g(t), h(t) >$

$$\begin{aligned}\int_a^b \vec{r}dt &= \left\langle \left(\int_a^b f(t)dt\right), \left(\int_a^b g(t)dt\right), \left(\int_a^b h(t)dt\right) \right\rangle \\ &= \vec{R}(b) - \vec{R}(a),\end{aligned}$$

where $\vec{R}\,'(r) = \vec{r}(t)$.

Examples

1. Find the unit tangent vector to to $\vec{r}(t) = \langle \sqrt{2t+2}, ln(t), t^2 \rangle$ at $t = 1$. Now find an equation for the tangent line to $\vec{r}(t)$ at $t = 1$.
2. If $\vec{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle$, find $\vec{T}(0)$, $\vec{r}\,''(0)$ and $\vec{r}\,'(t) \cdot \vec{r}\,''(t)$.
3. Evaluate $\displaystyle\int \left\langle te^{2t}, \frac{1}{1-t}, \frac{1}{\sqrt{1-t^2}} \right\rangle dt$.
4. Find $\vec{r}(t)$ if $\vec{r}\,'(t) = \langle 2t, e^t, \sin(t) \rangle$ and $\vec{r}(0) = \langle 0, 1, -1 \rangle$.