Arc Length and Curvature

Length of a curve

Let $\vec{r}(t) =< f(t), g(t), h(t) >$, with $a \leq t \leq b$, and traversed only once.

The length of the curve is given by

$$\begin{aligned} L &= \int_a^b \sqrt{[f\,'(t)]^2 + [g\,'(t)]^2 + [h\,'(t)]^2}\, dt \\ &= \int_a^b |\vec{r}\,'(t)|\, dt \end{aligned}$$

Example: A particle travels along a curve with position vector $\vec{r}(t) = \langle \sin(2t), \cos(2t), t \rangle$. Compute the distance that the particle travels in the time interval $2 \leq t \leq 4$. $\vec{r}\,'(t) = \langle 2\cos(2t), -2\sin(2t), 1 \rangle$ Thus,

$$L = \int_2^4 \sqrt{4\cos^2(t) + 4\sin^2(t) + 1}\, dt = 2\sqrt{5}$$

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The arc length function is given by

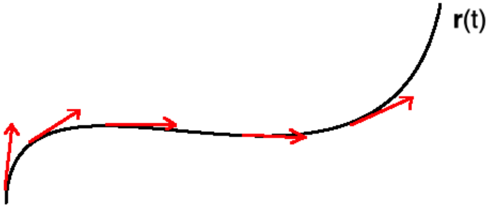
$$l(s) = \int_a^s \sqrt{[f\,'(t)]^2 + [g\,'(t)]^2 + [h\,'(t)]^2}\, dt = \int_a^s |\vec{r}\,'(t)|\, dt$$

($l(s)$ is the length of the curve from $\vec{r}(a)$ to $\vec{r}(s)$ and is independent of the parameterization used for the curve)
So

$$\frac{dl}{dt} = |\vec{r}\,'(t)|.$$

Curvature

The curvature is a measure of how quickly a curve $\vec{r}(t)$ changes direction - tangent vectors (or how they change) are useful for this purpose.



Curvature

The curvature is a measure of how quickly a curve $\vec{r}(t)$ changes direction - tangent vectors (or how they change over a distance) are useful for this purpose.

Recall: the unit tangent vector is $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

Definition

The **curvature** of a curve $\vec{r}(t)$ is

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{\left|\vec{T}\,'(t)\right|}{|\vec{r}\,'(t)|}$$

Theorem

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

If $y = f(x)$ then $\vec{r}(t) = \langle x, f(x) \rangle$ is a parameterization. Then

$$\kappa(x) = \frac{|f\,''(x)|}{(1 + (f\,'(x))^2)^{3/2}}$$

Examples

1. Find the arc length of $\vec{r}(t) = \langle \cos(t), \sin(t), ln(\cos(t)) \rangle$, for $0 \leq t \leq \pi/4$
2. Find the curvature of $\vec{r}(t) = \langle t^2, ln(t), t\, ln(t) \rangle$ at the point $(1, 0, 0)$
3. Find the curvature of $y = xe^x$.
4. In decreasing order, give the points with the greatest curvature: