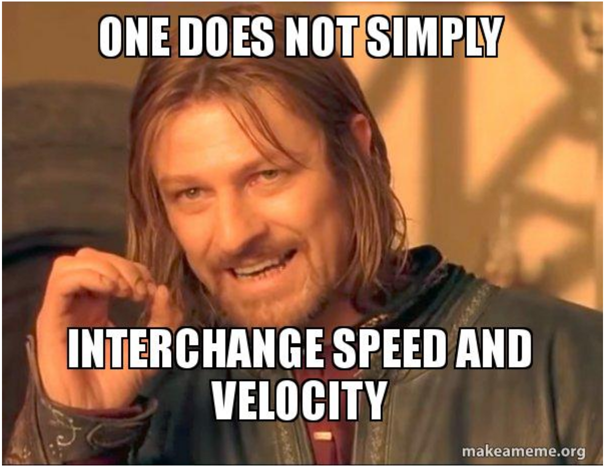
Motion in Space - velocity & acceleration



Velocity and acceleration

Suppose a particle moves through space with position vector $\vec{r}(t)$ at time t. Then, its

1. **velocity** is: $\vec{v}(t) = \vec{r}\,'(t)$,
2. **speed** is: $|\vec{v}(t)|$ $\quad (= \frac{ds}{dt}$ rate of change of distance w.r.t. time)
3. **acceleration** is: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

Application

Newton's second law of motion:

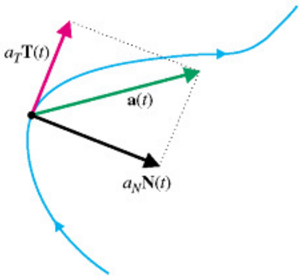
$$\vec{F}(t) = m\vec{a}(t)$$

Example: A particle moves along a curve with position vector $\overrightarrow{r}(t) = \langle e^t, ln(t), t^2 + 2t\rangle$. Find its velocity and acceleration. What is its speed at time $t = 1$?

Normal and tangential components of acceleration

It is sometimes useful express an objects acceleration as the sum of two vectors:

1. in the direction of its motion - i.e. a **tangential component**, and
2. in a direction perpendicular to to its motion (or its tangent) - i.e. a **normal component**



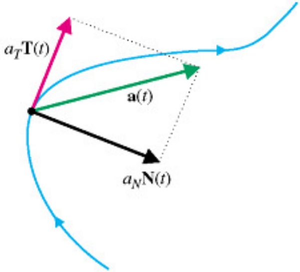
Normal and tangential components of acceleration

Let $\vec{r}(t)$ be the position vector of a moving object at time t.

The **unit tangent vector** is $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{v}}{|\vec{v}|}$.

Define the **unit normal vector** to be $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$ (this vector points in the direction the object is turning)

So $\vec{a} = a_T\vec{T} + a_N\vec{N}$ (can see this by the parallelogram law in the sketch below - we'll describe the scalar functions in the next slide)



Let's describe a_T and a_N:

We can express the acceleration of a moving object as

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

where the **tangential components** is

$$a_T = |\vec{v}|' \text{ (i.e. the derivative of its speed)}$$

and the **normal component** is

$$a_N = \kappa |\vec{v}|^2 \text{ (i.e. curvature times speed squared)}$$

So

$$\vec{a} = \left(|\vec{v}|'\right) \vec{T} + \left(\kappa |\vec{v}|^2\right) \vec{N}$$

Alternatively,

$$a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|} = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}$$

and

$$a_N = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

Examples

1. A particle moves along the curve $\vec{r}(t) = \langle e^{2t-4}, \ln(t), \sqrt{t} \rangle$. (a) Find its velocity, speed and acceleration at time $t = 2$. (b) What is the vector equation of the tangent line to $\vec{r}(t)$ at time $t = 2$.
2. A particle moves through space and has acceleration $\vec{a}(t) = \langle e^t, t^2, \cos(t) \rangle$. Find its position vector $\vec{r}(t)$ if $\vec{r}(0) = \langle 2, 1, 0 \rangle$ and $\vec{v}(0) = \langle 1, 0, 0 \rangle$.
3. (Example 4 in Steward, p.912) A particle moves in a circular path with constant angular speed ω and has position vector $\vec{r}(t) = \langle \cos(\omega t), \sin(\omega t) \rangle$. Find the force $\vec{F}(t)$ acting on the particle.
4. Find the tangential and normal components of the acceleration of $\vec{r}(t) = \langle t, 2e^t, e^{2t} \rangle$ at time $t = 0$.