# Sequences and Series

If

$$\lim x \to 8 \quad \frac{1}{x - 8} = \infty$$

then

$$\lim x \to 5 \quad \frac{1}{x - 5} = \rotatebox{90}{5}$$

# Sequences

## Definition

An infinite **sequence** of numbers is a list of numbers $(a_1, a_2, a_3, \dots)$, which is often written as $(a_n)_{n=1}^{\infty}$ or $\{a_n\}_{n=1}^{\infty}$.

Examples:

- $(2, 2, 2, 2, 2, \dots)$ is a constant sequence.
- $(1, -1, 1, -1, \dots)$ or $\{(-1)^n\}_{n=0}^{\infty}$ or $a_n = (-1)^n$.
- $(1, 3, 5, 7, \dots)$ or $(2n+1)_{n=0}^{\infty}$ or $a_n = 2n + 1$.
- $\left(\frac{1}{n}\right)_{n=1}^{\infty}$.

# Sequences

## Definition

- A sequence $(a_n)_{n=1}^{\infty}$ **converges** to number a $L$, if for ever $\epsilon > 0$ there is a number $N \in \mathbb{N}$ such that

$$\text{whenever } n > N \text{ then } |a_n - L| < \epsilon.$$

(we call $L$ the **limit** and write $\lim_{n\to\infty} a_n = L$.)

- The sequence **diverges** if it does not converge.

Example: Use the definition of convergence to show that

$$\lim_{n\to\infty} \frac{4n^2 + n - 1}{n^2 + 1} = 4.$$

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We want to show that $\left|\frac{4n^2+n-1}{n^2+1} - 4\right| \le \epsilon$. Note that

$$\begin{aligned} \left|\frac{4n^2+n-1}{n^2+1} - 4\right| &= \left|\frac{n-5}{n^2+1}\right| \\ &= \frac{n-5}{n^2+1} \qquad \text{(as long as } n \ge 5) \\ &\le \frac{1}{n} \qquad (\text{ check!}) \end{aligned}$$

So we choose $N$ such that $N > \frac{1}{\epsilon}$ (i.e. $\frac{1}{N} < \epsilon$) and $N \ge 5$. Now, if $n \ge N$ (i.e. $\frac{1}{n} \le \frac{1}{N}$), then

$$\left|\frac{4n^2+n-1}{n^2+1} - 4\right| \le \frac{1}{n} \le \frac{1}{N} < \epsilon.$$

Sometimes we can say *how* a sequence diverges:

**Definition**

- $\lim_{n\to\infty} a_n = \infty$ (diverges to infinity), if for every $M$, there is $N$ such that for all $n > N$ we have $a_n > M$.
- $\lim_{n\to\infty} a_n = -\infty$ (diverges to negative infinity), if for every $M$, there is $N$ such that for all $n > N$ we have $a_n < M$.

# Useful limits

1. $$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 1 & \text{if } r = 1 \\ 0 & \text{if } |r| < 1 \\ \text{no limit} & \text{if } r \leq -1 \end{cases}$$

2. $$\lim_{n\to\infty} \frac{c^n}{n!} = 0$$

3. $$\lim_{n\to\infty} \frac{n^n}{n!} = \infty$$

4. $\lim_{n\to\infty} T_n(x)$ where $T_n(x)$ is the Maclaurin polynomial of $f(x) = \frac{1}{1-x}$.
   Here are some steps:

   (i) From the preliminary homework $T_n(x) = 1 + x + x^2 + \cdots + x^n$. Now show $T_n(x) = \dfrac{1 - x^{1+n}}{1 - x} = f(x) - \dfrac{x^{1+n}}{1 - x}$.

   (ii)Therefore $R_n(x) = \dfrac{x^{1+n}}{1 - x}$

   (ii) Now use (1) to determine when $\lim_{n\to\infty} T_n(x) = f(x)$.

# Examples

In each case find the limit if it exists.

1. $\lim_{n\to\infty} \frac{c}{n}$ ($c$ is any constant)
2. $\lim_{n\to\infty} \frac{3n^4 + 5n^2 - n + 1}{n^4 - n + 1}$
3. $\lim_{n\to\infty} \frac{n^2 - 3n}{e^n + 4n}$
4. $\lim_{n\to\infty} \frac{3^n}{2^{n+4}}$
5. $\lim_{n\to\infty} \frac{3^{2n} - n!}{5^n}$