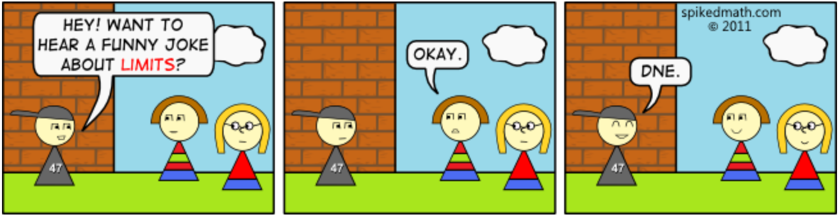
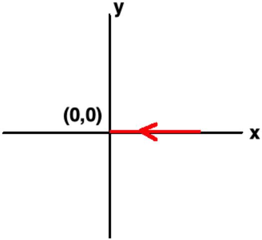
Limits and Contuity



Consider $f(x,y) = \frac{x^2+y^2}{x^2+2y^2}$ and the limit $\lim_{(x,y)\to(0,0)} f(x,y)$.

Let's see what happens if we approach $(0,0)$ from two paths: the positive x-axis and then the positive y-axis.

Along the x-axis $y = 0$,



Then

$$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$

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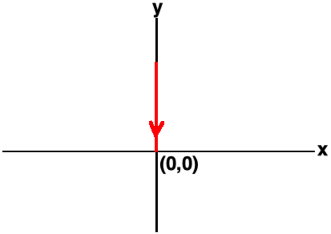

Then

$$\lim_{(0,y)\to(0,0)} f(0,y) = \lim_{(0,y)\to(0,0)} \frac{y^2}{2y^2} = \frac{1}{2}$$

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The limits differ when approached from two different paths!!
Hence

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

cannot exist; else it would be ambiguous - is it 1 or $\frac{1}{2}$?!

- **Moral**: For functions of more that one variable there are infinitely many paths from which to approach a limit
  - So the limits exists if ALL of the infinitely many directions give the SAME limit.
- **Upshot**: To show a limit **DNE** we only need to show the limits approached from two distinct paths differ.

Limit

Let $f : D \to \mathbb{R}$ be a function with $D \subseteq \mathbb{R}^2$. Then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$, whenever $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Continuous

A function $f : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}^2$, is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Examples

Find the limit, if it exists, and determine if the function is continuous at the limiting point.

1. $\lim_{(x,y)\to(2,\pi)} e^{xy} - x\cos y - \dfrac{2x^3}{y}$,
2. $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$,
3. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$,
4. $\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^4+3y^4}$,
5. $\lim_{(x,y)\to(0,0)} \frac{x^4y}{x^8+y^4}$,
6. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$