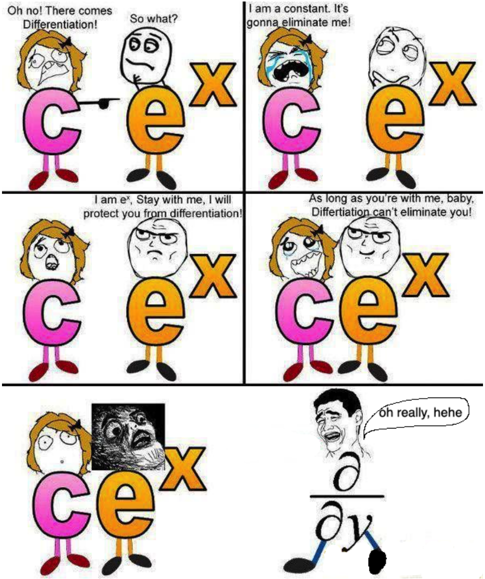
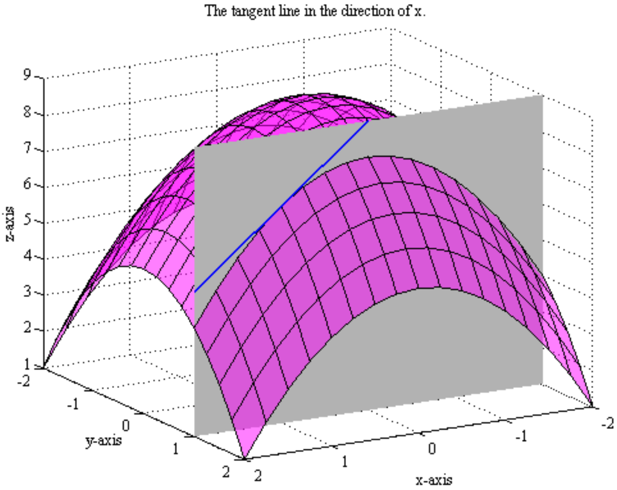
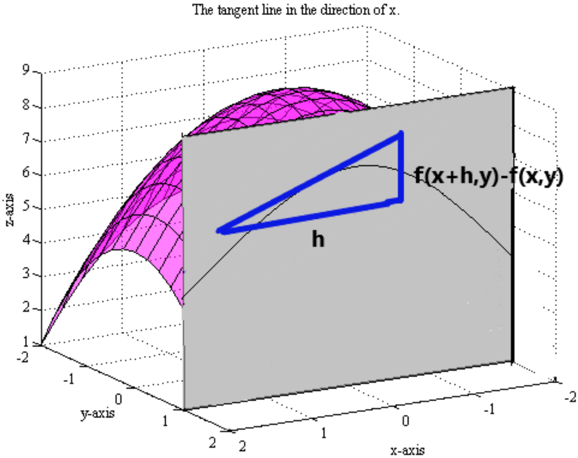
Partial derivatives





The tangent line in the direction of x.
z-axis
9
8
7
6
5
4
3
2
1
-2
-1
0
y-axis
1
2
2
1
0
-1
-2
x-axis



The tangent line in the direction of x.
f(x+h,y)-f(x,y)
h
z-axis
y-axis
x-axis

Let $f(x,y)$ be a two variable function. Then the partial derivatives are

Definitions

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \end{aligned}$$

A partial derivative is a derivative in the x direction or the y direction. Moral: find the **usual derivative w.r.t. one variable** by viewing all other variables as constants.

Definitions

Notation

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_1 f = D_x f.$$

(Similarly for other variable.)
Higher derivatives:

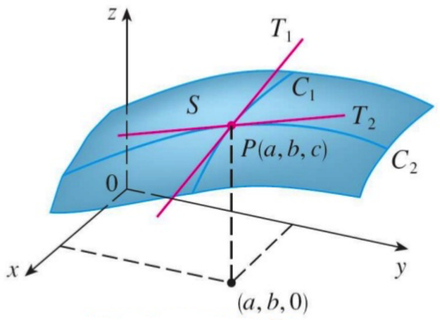
$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

(similarly for $f_{xx}, f_{xyy}, f_{xyzyxzzzzyx}$ etc.)

Interpretations

We can interpret f_x, for example,

- as the **rate of change** of $z = f(x, y)$ as x changes
- geometrically: $f_x(a, b)$ is the **gradient/slope** of the tangent to curve where the plane $y = b$ intersects the surface $f(x, y)$.

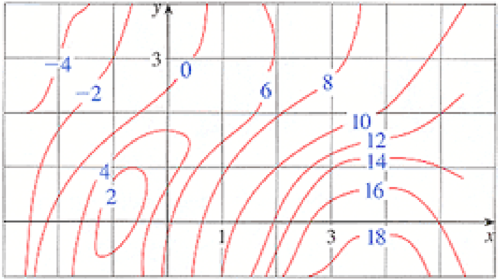


Clairaut's Theorem

Suppose $f(x, y)$ is defined on a disk D that contains the point (a, b). If the functions $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Use the contour map below to estimate $f_x(2,1)$ and $f_y(2,1)$.



Examples

Find f_x, f_y, f_{xx}, f_{xy} and f_{xxxy}

1. $f(x, y) = \cos(x) + e^y$
2. $f(x, y) = \ln(xy) - x^2y^3$
3. $f(x, y) = \sin(x^2 - 5y) - xe^xy$
4. $f(x, y) = \sqrt{\sin(x) - 4y} + (x + 2y)^2$

Examples of applications

1. Laplace's equation
$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 x} = 0$$
models fluid flow, electrostatics, steady-state heat conduction.
Show that $u(x, y) = e^y \cos(x)$ **satisfies the Laplace equation.**
2. Wave equation
$$\frac{\partial^2 u}{\partial^2 t} = a^2 \frac{\partial^2 u}{\partial^2 x}$$
models waves (e.g. $u(x, t)$ could represent the amplitude of the wave a distance x from some point and at time t).
Show that $u(x, t) = \cos(x - at) + \sin(x - at)$ **satisfies the wave equation.**