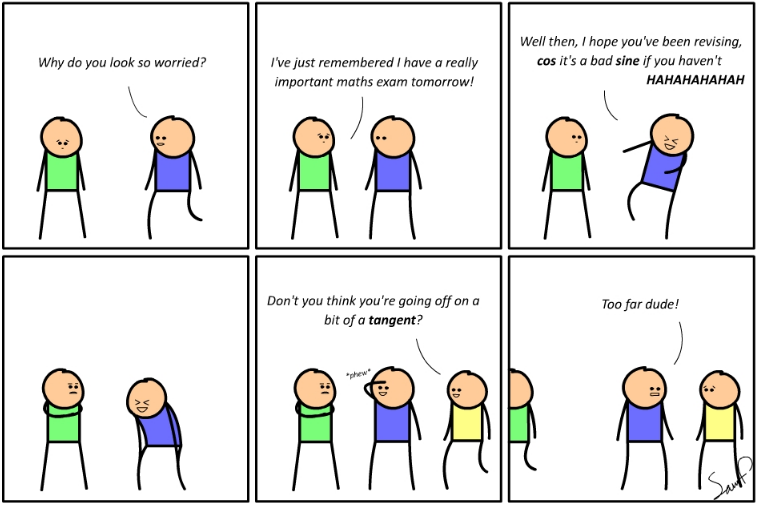
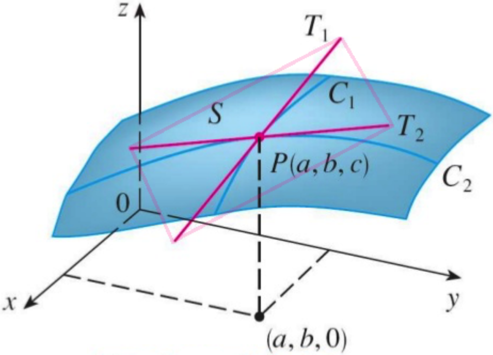
Tangent Planes and Linear Approximations



If $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative at $x = a$, then (for b "close" to a) we can approximate $f(b)$ with the equation of the tangent line at $f(a)$.

Analogously,

If $f : \mathbb{R}^2 \to \mathbb{R}$ has a continuous *partial* derivatives, then we want to approximate f with a tangent *plane*.



The **tangent plane** of $f(x, y)$ at $P(a, b, c)$ is defined as the plane containing both tangent lines T_1 and T_2 (having directions f_x and f_y and with the point P in common)

Tangent plane

If $f(x, y)$ has continuous partial derivatives, then the equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Find the equation of the tangent plane to the graph of $f(x, y) = \sin(xy)$ at $(2, \pi, 0)$.

Solution: $f_x(2, \pi) = \pi$ and $f_y(2, \pi) = 2$. Thus,

$$z - 0 = \pi(x - 2) + 2(y - \pi),$$

or

$$\pi x + 2y - z = 4\pi$$

is the equation of the tangent plane

Linear apprximation

Consider a function $z = f(x, y)$ with continuous partial derivatives at a point $(a, b, f(a, b))$. Then the equation of the tangent plane to f at $(a, b, f(a, b))$ is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

or

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

(so z is a linear function in the two variables x and y)

The **linearization** of f at (a, b) is the function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The **linear aproximation** (or tangent approximation) of f at (a, b) is

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Examples

1. Find the equation of the tangent plane to the graph of $f(x, y) = \ln(xy) - 3y^2$ at the point $(1, 1, -3)$.
2. Find the linearization of $f(x, y) = x^2 e^y + \sin\left(\frac{y}{x}\right)$ at $(1, 0)$.
3. Use a linear approximation of $f(x, y) = x^2 e^y + \sin\left(\frac{y}{x}\right)$ to estimate $f(1.1, -0.1)$.