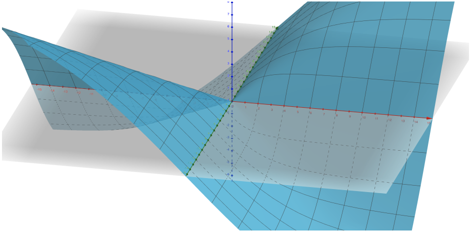
Tangent Approximations and Differentiability

- We want to define the notion of **differentiability** for functions of two variables.
- It is tempting to say $f(x, y)$ is differentiable at (x_0, y_0) if the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are exist and are defined.
- However, with this definition we run into some issues.
  For example, it is not compatible with the idea of using the partial derivatives to find a tangent plane and using this plane as a linear approximation.

For example, consider $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$.



- It can be shown that both partial derivatives are defined at $(0, 0)$. Tangent plane at $(0, 0)$?
- A problem is that the partial derivatives only give a rate of change (or a derivative) in two very specific directions - what about the other infinitely many directions?

A way to remedy this problem is to *define* differentiability as having a (non-vertical) tangent plane.
Then we just need to sort out what it means to *be a tangent* plane.

Idea: apply the notion of a slope - how the function values change as the distance from the point (x_0, y_0) changes:

$$\frac{f(x,y) - f(x_0,y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

If a tangent plane P is to approximate a given function $f(x,y)$ (in all directions), we want the difference in slopes be as small as we like.
More precisely,

Definition

P is tangent to f at (x_0, y_0, z_0) if and only if $f(x_0, y_0) = P(x_0, y_0)$ and

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - P(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Definition

A function $f : \mathbb{R}^2 \to \mathbb{R}$ is **differentiable** at (x_0, y_0) if its graph has a non-vertical tangent plane at (x_0, y_0). This tangent plane is unique with equation

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$