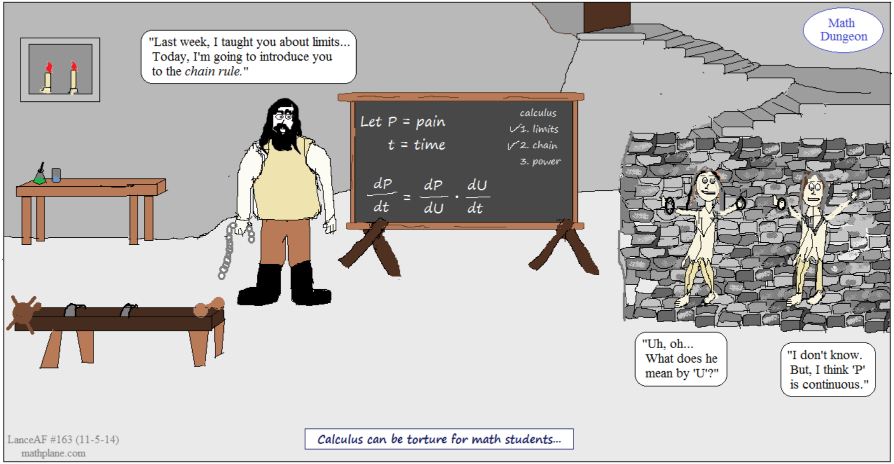
Chain Rule



Math Dungeon
"Last week, I taught you about limits... Today, I'm going to introduce you to the *chain rule*."
Let P = pain
t = time
calculus
1. limits
2. chain
3. power
dP/dt = dP/dU · dU/dt
"Uh, oh... What does he mean by 'U'?"
"I don't know. But, I think 'P' is continuous."
Calculus can be torture for math students...
LanceAF #163 (11-5-14)
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For $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$, and all functions are differentiable:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example: Find $\frac{dz}{dt}$ if $z = 3x + y\cos(x)$, where $x = 5t^2 - 1$ and $y = \sin(t)$.

Solution: We find the various derivatives (in terms of t):

$\frac{\partial z}{\partial x} = 3 - y\sin(x) = 3 - \sin(t)\sin(5t^2 - 1)$ and

$\frac{\partial z}{\partial y} = \cos(x) = \cos(5t^2 - 1)$ and $\frac{dx}{dt} = 10t$, $\frac{dy}{dt} = \cos(t)$

Thus,

$$\frac{dz}{dt} = (3 - \sin(t)\sin(5t^2 - 1))10t + \cos(5t^2 - 1)\cos(t)$$

For $z = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$, and all functions are differentiable:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: Find $\frac{\partial f}{\partial s}$ if $f(x, y) = xe^y$, where $x(s, t) = s + 2t$ and $y(s, t) = sln(t)$.

Solution: $\frac{\partial z}{\partial s} = e^{sln(t)} + (s + 2t)e^{sln(t)}ln(t)$

Implicit functions

For $F(x, y, z) = 0$, where $z = f(x, y)$ implicitly:

$$\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial F}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)}$$

Examples

1. $z = x^2 - y^2 + 4xy$, with $x = 3\cos(t)$ and $y = e^t$. Find $\dfrac{dz}{dt}$.
2. $z = \sin(x)\cos(y)$, with $x = \frac{1}{t^2}$ and $y = ln(t)$. Find $\dfrac{dz}{dt}$.
3. $z = e^{(x^2 - \cos(y))}$, with $x = st^2$ and $y = te^s$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.
4. $z = x\sin(y) - xy$, with $x = \frac{u}{v^2}$ and $y = \tan(uv)$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.
5. If $x^3 - 2y = \sin(xy)$, find $\dfrac{dy}{dx}$.
6. If $ye^x + ln(x + y) = x\cos(y)$, find $\dfrac{dy}{dx}$.