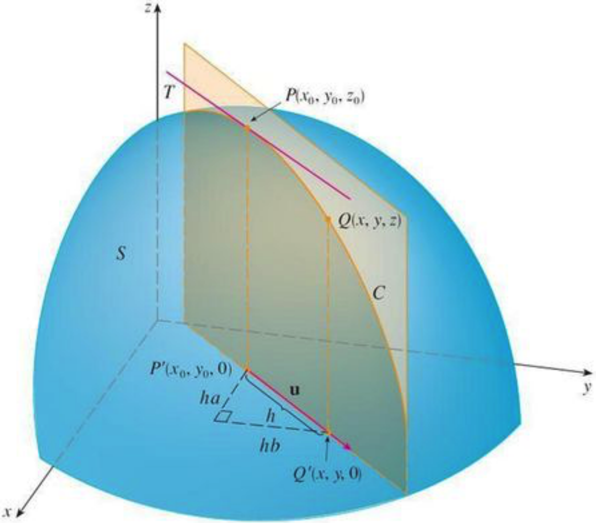
Directional derivatives


z
T
P(x0, y0, z0)
Q(x, y, z)
S
C
P′(x0, y0, 0)
u
ha
h
hb
Q′(x, y, 0)
y
x

Definitions

Let $f(x, y)$ be a two variable function.

The **directional derivative** of $f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.

Theorem

If $f(x, y)$ is differentiable, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Gradient vector

Note:

$$D_{\mathbf{u}}f = f_x a + f_y b = \langle f_x,\ f_y \rangle \cdot \langle a,\ b \rangle$$

Definition

The **gradient** of $f(x, y)$ is the vector function

$$\nabla f(x, y) = \langle f_x(x, y),\ f_y(x, y) \rangle$$

In this notation the directional derivative can be written as

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \langle a,\ b \rangle$$

Example: Find the derivative of $f(x, y) = x^2 ln(y)$ at $(3, 1)$ in the direction $\vec{u} = 3\vec{i} - 4\vec{j}$.

Consider $f(x, y, z)$ and a unit vector $\mathbf{u} = \langle a, b, c \rangle$. Then

$$D_{\mathbf{u}} f = f_x a + f_y b = f_z c = \langle f_x,\ f_y,\ f_z \rangle \cdot \langle a,\ b,\ c \rangle$$

Definition

The **gradient** of $f(x, y, z)$ is the vector function

$$\nabla f(x, y, z) = \langle f_x(x, y, z),\ f_y(x, y, z),\ f_z(x, y, z) \rangle$$

Hence,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y, z) \cdot \langle a,\ b,\ c \rangle$$

What is the maximum rate of change of f, and in which direction does this ocur?

Theorem

If f is differentiable, then the maximum value of $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\mathbf{u}$ has the same direction as $|\nabla f|$.

Tangent planes to level surfaces

Suppose S is a surface with equation $F(x, y, z) = k$.

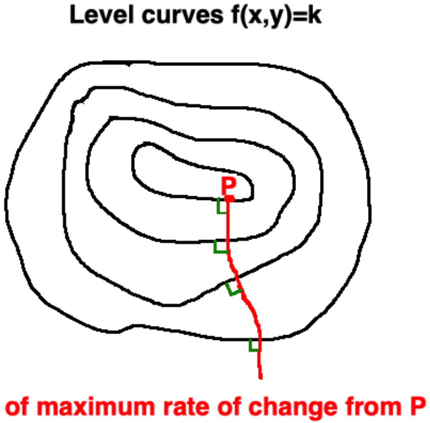
The **tangent plane to** S at the point (x_0, y_0, z_0) has normal vector

$$\nabla F(x_0, y_0, z_0) = \langle F_x(x_0, y_0, z_0),\ F_y(x_0, y_0, z_0),\ F_z(x_0, y_0, z_0)\rangle$$

and equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

For a level curve (or level surfaces) ∇f is always perpendicular to the (tangent of) the curve; that is, **the maximum rate of change occurs in a perpendicular direction to a level curve (or surface)**



Examples

(a) Find the directional derivative at the given point, the maxim rate of change and the direction in which it occurs:

1. $f(x,y) = \dfrac{e^x}{4y}$ at $(0,2)$ in the direction $\mathbf{v} = \langle 1,1 \rangle$
2. $F(x,y,z) = ln(zy) - x\cos(y) + z$ at $(1,\pi,2)$ in the direction $\mathbf{v} = \langle -1,0,2 \rangle$

(b) Find the equation of the tangent plane to the surface given by $xe^y - xz^2 = -9$ at the point $(3,0,-2)$