Taylor Series

Series: *exist*

Everyone:

Series

Definitions

- An infinite **series** is a *sum* of infinitely many terms

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

- The n^{th} **partial sum** is the sum of the first n terms

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots a_n$$

- To give meaning to an infinite series we define the sum (i.e. the value it adds up to) to be

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

Taylor series

So $\sum_{k=1}^{\infty} a_k$ is the limit of the *sequence* of partial sums $S_1, S_2, S_3, \ldots$.

If the sequence $\{S_n\}_{n=1}^{\infty}$ converges (i.e. $\lim_{n\to\infty} S_n = L$), then we say the **series** $\sum_{k=1}^{\infty} a_k$ **converges**, and it **diverges** otherwise.

The same idea can be applied to the sequence of Taylor polynomials $T_n(x) = \sum_{k=0}^{n} \dfrac{f^{(k)}(a)}{k!}(x-a)^k$ of a function f:

Definition

The **Taylor series** of a function f centered at $x = a$ is the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

- We've seen that the Taylor series of some functions converge for certain values of x and diverge for others; e.g. Maclaurin series of $\frac{1}{1-x}$ converges for $|x| < 1$.

- Next we'll look at some tests to determine when certain series converge, with the goal of applying it to Taylor series later on.

- You may accept (without proof) that the Taylor series for $\sin(x), \cos(x)$ and e^x converge to its function for all values of x.

- We'll also see that sometimes the best we can do is say if a series converges or not (without finding the sum). However, for a Taylor series this is enough as we already know the function and only want the x-values for which it converges!

Convergent series

Definion

A series $\sum_{k=1}^{\infty} a_n$ **converges absolutely** if $\sum_{k=1}^{\infty} |a_n|$ converges.

Proposition

If $\sum_{k=1}^{\infty} a_n$ converges absolutely, then it converges. Converse? - No, e.g. $\sum_{n=0}^{\infty} \frac{1}{n}$

Convergent series

Proposition

Suppose $\sum_{k=1}^{\infty} a_n$ and $\sum_{k=1}^{\infty} b_n$ are such that $0 \leq a_n \leq b_n$ for all n (terms may not be negative!). If $\sum_{k=1}^{\infty} b_n$ converges, then $\sum_{k=1}^{\infty} a_n$ converges.

Proposition

If $\sum_{k=1}^{\infty} a_n$ converges absolutely, then it converges. Converse? - No, e.g. $\sum_{n=1}^{\infty} \frac{1}{n}$

Test for divergence

If $\sum_{k=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ (So, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{k=1}^{\infty} a_n$ diverges)

Geometric series

Definion

A series $\sum_{n=0}^{\infty} ar^n$, with $a \neq 0$, is called a **geometric series** with ratio r.

We've seen such a series before - the Maclaurin series for $\dfrac{1}{1-x}$, and we know when it converges!

Propoistion

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Examples:

$$(i) \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n} = 6 \left(|r| = \frac{2}{3} < 1 \right) \qquad (ii) \sum_{n=0}^{\infty} \frac{5^n}{4^n} = \text{div} \left(|r| = \frac{5}{4} > 1 \right)$$

Alternating series

Definion

A series $\sum_{n=0}^{\infty}(-1)^n a_n$, with $a_n > 0$ for all n, is called an **alternating series** (i.e. the terms have alternating signs).

We've seen such a series before too - the Maclaurin series for $\cos(x)$.

Alternating series test

An alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ converges if

- $a_{n+1} \leq a_n$, and
- $\lim_{n\to\infty} a_n = 0$

Examples:

$$(i) \sum_{n=2}^{\infty}(-1)^n \frac{2n+1}{n^2-n} = \text{conv} \qquad (ii) \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n} = \text{conv}$$

Examples

Determine if the following series are convergent, absolutely convergent or divergent, and find the sum if possible.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{7^{n+1}}{10^n}$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{3n^2 - 4}$
5. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{6^n + 1}$
6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{n^2 + 1}$