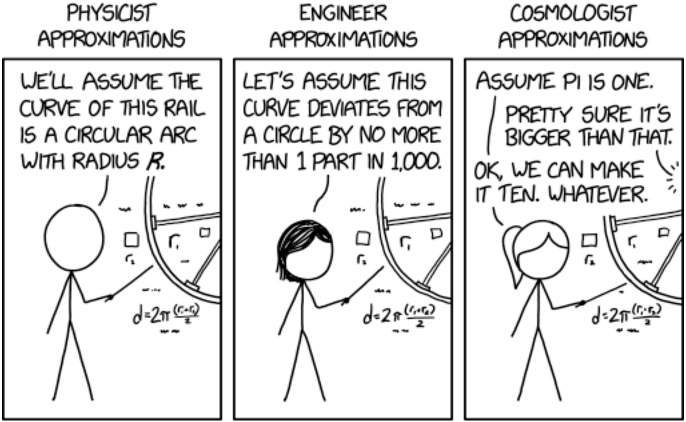
Error Estimates



Error

- An important application of series is estimating a function by using a partial sum S_n of the series (e.g. estimating e^2)
- Depending on the application, a certain degree of accuracy is needed. So we need to know how many terms to add up so that the error is small enough for our needs.

Definitions

The **error** in estimating a series $\sum_{k=0}^{\infty} a_n$ by its n^{th}-partial sum is given by

$$|R_n| = \left|\sum_{k=0}^{\infty} a_k - S_n\right| = \left|\sum_{k=n+1}^{\infty} a_k\right|$$

Terminology: *Bounding the error* means finding a number b such that the error is guaranteed to be smaller than b; that is, $\left|\sum_{k=n+1}^{\infty} a_k\right| < b$.

Alternating series error

Let $\sum_{n=0}^{\infty} (-1)^n a_n$ be an alternating series. Then

$$|R_n| = \left| \sum_{k=n+1}^{\infty} a_k \right| \le a_{n+1}$$

Example: Find n such that $e^{-1} \approx \sum_{k=0}^{n} \frac{(-1)^n}{k!}$ is correct up to 3 decimal places:

$$|R_n| \le a_{n+1} = \frac{1}{(n+1)!}$$

To find n, we should have $\frac{1}{(n+1)!} < 0.0005 = \frac{5}{10000} = \frac{1}{2000}$. That is, $(n+1)! > 2000$. Computing factorials we see that $6! = 720$ and $7! = 5040$. So $n = 6$ will do the job!

Bounding by comparison

Examples: Find n such that $e \approx \sum_{k=0}^{n} \frac{1}{k!}$ is correct up to 3 decimal places: Need n such that $\left|\sum_{k=n+1}^{\infty} a_k\right| \leq 0.0005$. Note:

- $a_k = \frac{1}{k!}$, which means that

$$a_{k+1} = \frac{1}{(k+1)!} = \frac{1}{k!}\frac{1}{(k+1)} = a_k \frac{1}{(k+1)}$$

- Since we consider a tail of the series (i.e. $\sum_{k=n+1}^{\infty} a_k$), we have $k+1 > n+1$ and therefore

$$a_{k+1} = a_k \frac{1}{(k+1)} < a_k \frac{1}{(n+1)}.$$

- Thus, our series is term by term less than the series with terms

$$a_{n+1}, a_{n+1}\frac{1}{(n+1)}, a_{n+1}\left(\frac{1}{(n+1)}\right)^2, a_{n+1}\left(\frac{1}{(n+1)}\right)^3, \ldots.$$

- This is a geometric series with ratio $\frac{1}{(n+1)}$ and thus converges to $\frac{1}{n!(n)}$.

Therefore solving $\frac{1}{n!(n)} \leq 0.0005$ gives n so that our error is less than 0.0005.

Examples

1. How many terms are needed to estimate $\sum_{k=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^k}{k}$ such that error< 0.05.
2. How many terms are needed to estimate $\sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)^k}{k}$ such that error< 0.05.
3. Give a partial sum that estimate $\sin\left(\frac{1}{2}\right)$ correct up to 2 decimal places.
4. Approximate the sum of the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$ correct up to 3 decimal paces