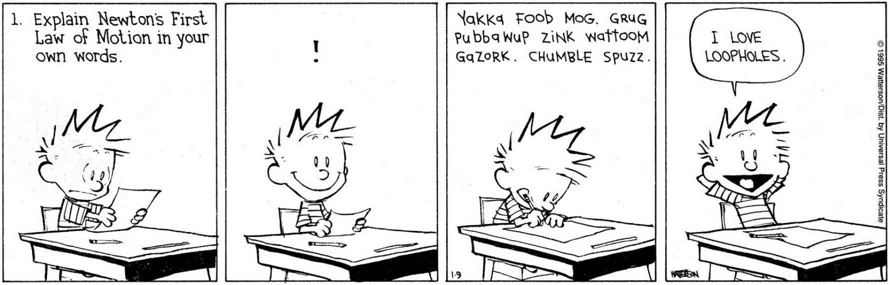
Power series



Ratio test

Proposition

Let $\sum_{n=0}^{\infty} a_n$ be any series and suppose

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

If

- $L < 1$ then $\sum_{n=0}^{\infty} a_n$ is (absolutely) convergent
- $L > 1$ then $\sum_{n=0}^{\infty} a_n$ is divergent
- $L = 1$ then this test says nothing....

Examples:

$$(i)\sum_{n=0}^{\infty}\frac{n!}{5^n}\ \text{-div} \qquad (ii)\sum_{n=0}^{\infty}\frac{2^n}{(n+1)!}\ \text{-conv}$$

Power series

Definition

A power series centered at $x = a$ has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots ,$$

where each c_n is a constant.

Note:
(i) a power series defines a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$,
(ii) every Taylor series is power series with $c_n = \dfrac{f^{(n)}(a)}{n!}$.

Power series convergence - example

We can use the ratio test to find the x-values for which a power series converges: Consider

$$\sum_{n=0}^{\infty}(-1)^n n 4^n x^n.$$

1. Write an expression for $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)4^{n+1}x^{n+1}}{n4^n x^n}\right|$
2. Now take the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left(\lim_{n\to\infty}\frac{(n+1)}{n}\right)4|x| = 4|x|$
3. For which value(s) of x is the limit above less that 1?
   When $|x| < \frac{1}{4}$ or $-\frac{1}{4} < x < \frac{1}{4}$
4. Does the series converge for $x = -\frac{1}{4}$ and/or $\frac{1}{4}$? Div for both

The interval $(-\frac{1}{4}, \frac{1}{4})$ is called the **interval of convergence** and $R = \frac{1}{4}$ the **radius of convergence**.

Convergence of power series - in general

For a general power series $\sum_{n=0}^{\infty} c_n(x-a)^n$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left(\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\right)|(x-a)|$$

Let $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$. Then $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely if

$$L|x-a|<1, \text{ that is, when } |x-a|<\frac{1}{L}$$

We call $R=\frac{1}{L}$ the **radius of convergence** and the **interval of convergence** is one of the following: $(a-R,a+R)$, $(a-R,a+R]$, $[a-R,a+R)$ or $[a-R,a+R]$.

Power series differentiation and integration

Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$. Then

1. $f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$,
2. $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$, and

in both cases the **radius of convergence is the same** as $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ (caution: the interval of convergence might change at the ends points!).

That is, differentiation and integration is done term-by-term in the series and doesn't change radius of convergence.

Power series representation, differentiation and integration - an application

Consider the function $g(x) = \int_0^x e^{-x^2}\,dx$. We can approximate $g(2)$ with power series:

1. Use the power series of e^x to find a power series for e^{-x^2}.
2. What is the radius and interval of convergence for e^{-x^2} and is 2 in this interval?
3. Now integrate e^{-x^2} to find $g(x)$.
4. How many terms are needed to appropriate $g(2)$ correct up to 2 decimal places?

Exercises

1. Find the radius and interval of convergence: $\sum_{k=1}^{\infty} \dfrac{x^k}{\sqrt{k}}$.
2. Find the radius and interval of convergence: $\sum_{k=1}^{\infty} \dfrac{x^k}{k3^k}$.
3. Find the radius and interval of convergence: $\sum_{k=0}^{\infty} \dfrac{x^n}{n!}$.
4. Find a power series representation for $\dfrac{3}{2+2x}$ (Hint: use the power series for $\frac{1}{1-x}$).
5. Find a power series representation for $\tan^{-1}(x)$ (Hint: use the derivative $\tan^{-1}(x)$ and its power series).