Applications fo Integration

Work

Work with constant force

When applying a **constant force** F over a distance d, the work done is

$$W = F \times d$$

Work with varying force

When the force applied depends on the distance (i.e. **force** $F(x)$ **is function of position** x), then the work done over a distance/interval $[a, b]$ is

$$W = \int_a^b F(x)\,dx$$

Example

When a particle is located a distance x meters from the origin, a force of $\cos(\frac{\pi x}{3})$ newtons acts on it. How much work is done by moving the particle from $x = 1$ to $x = 2$?

(a) First apprroximate the work with a Riemann sum. Riemann sum:

$$W \approx \sum_{i=1}^{n} F(x_i^*)\Delta x = \frac{1}{n}\sum_{i=1}^{n} \cos\left(\frac{\pi x_i^*}{3}\right),$$

where $\Delta x = \frac{2-1}{n}$ and $x_i^* \in [x_i, x_{i+1}]$.

(b) Now take a limit of your Riemann sum to get the exact work as an integral. Integral:

$$W = \int_1^2 \cos\left(\frac{\pi x}{3}\right)\, dx$$

Probability

Consider the question: what is the probability that a battery will last between 100 and 200 hours? Let X represent the lifetime of such a battery.

- The possibilities forms a continuous interval $[100, 200]$.
- We call X a **continuous random variable**.
- We denote the probability by $P(100 \leq X \leq 200)$.
- Every continuous random variable has **probability density function** $f(x)$

Probability density function

A probability density function for X is function such that

$$P(100 \leq X \leq 200) = \int_{100}^{200} f(x)\, dx$$

That is, the **probability for an interval is the area under the graph** of f between the interval's endpoints.

Example

Let $f(x) = 30x^2(1-x)^2$ for $0 \le x \le 1$ and let $f(x) = 0$ otherwise.
(a) Verify that f is a probability density (Hint: Probability of all possibilities is one)
(b) Give a Riemann sum that approximates $P(X \le \frac{1}{6})$
Riemann sum:

$$P(X \le \frac{1}{6}) \approx \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{6n}\sum_{i=1}^{n} 30(x_i^*)^2(1-x_i^*)^2,$$

where $\Delta x = \frac{\frac{1}{6}-0}{n}$ and $x_i^* \in [x_i, x_{i+1}]$.
(c) Now take a limit of your Riemann sum to get $P(X \le \frac{1}{6})$: Integral:

$$P(X \le \frac{1}{6}) = \int_0^{\frac{1}{6}} 30x^2(1-x)^2\, dx$$

Mass concentration

Constant density

The density of a 2-D material (like a wire) is a measure of how much material is present in a certain length - i.e.

$$\rho = \frac{mass}{length}$$

Example: metal wire with length 2 has a density of $\rho = 4\ lb/cm$. What is its mass? mass= $\rho\times$length= $4 \times (2)\ lb$.

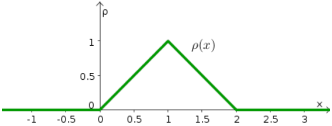
Varying density

Sometimes the density depends on a certain position (i.e. **density** $\rho(x)$ **is function of position** x), then the total mass over an interval $[a, b]$ is

$$mass = \int_a^b \rho(x)\, dx$$

The total mass is the area under the graph of ρ.

1. A 200 lb rope that is 100 ft long hangs over edge of a building. How much work is done pulling the rope to the top of the building.
2. Consider the function below



(a) Explain why ρ is a probability density function.
(b) Compute $P(X < 1)$.
3. The probability function for a customer's call to be answered by a call center is $f(t) = 0.2e^{-\frac{t}{5}}$ if $t \geq 0$ and zero otherwise. Find the probability that the customer waits more than 5 minutes.