Math 8 - Lecture 2 Nadia Lafrenière

Sequences and series 01/08/2020

I- Sequences

Question: What is the pattern of the sequence

$$1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots ?$$

This is $\frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot \dots \cdot n}$, with $0! = 1$, for $n \in \{0, 1, 2, \dots\}$.

Definition

A sequence is a function $f: \underbrace{\mathbb{N}}_{0,1,2,\dots} \to \mathbb{R}$.

Real numbers.

We write it as $(f_n)_{n \in \mathbb{N}} = (f_n)_{n=0}^{\infty}$ (or, in the textbook, as $\{f_n\}_{n=0}^{\infty}$).

Example

$\left(\frac{1}{n!}\right)_{n \in \mathbb{N}}$ is the sequence whose first terms are above.

Example

In the preliminary homework, you computed the Maclaurin polynomial for $f(x) = \frac{1}{1-x}$. Then, the sequence $(T_n(x))_{n \in \mathbb{N}} = \left(\sum_{k=0}^{n} x^k\right)_{n \in \mathbb{N}}$ is the sequence of MacLaurin polynomials for $f(x)$ as n changes.

Definition

The limit of the sequence, if it exists, is a real number L such that, for all $\varepsilon > 0$, there exists a number N such that

$$|a_n - L| \leq \varepsilon, \text{ for all } n \geq N.$$

Then, we say $(a_n)_{n \in \mathbb{N}}$ converges to L.

Note that N depends on ε. Just imagine someone else is fixing a very small ε, and you need to take N accordingly so the statement is true.

Question: Which of these have limits?

a) $\left(\frac{1}{n!}\right)_{n \geq 1}$ b) $\left((-1)^n\right)_{n \in \mathbb{N}}$ c) $\left(\frac{(-1)^n}{n}\right)_{n \geq 1}$ d) $(n)_{n \in \mathbb{N}}$

Answer

a) and c) converge to 0

b) and d) diverge.

Arguments

a) For all $n \geq 1$, $0 \leq \frac{1}{n!} \leq \frac{1}{n}$ and $\lim_{n \to \infty} \frac{1}{n} = 0$

By squeezing, $\lim_{n \to \infty} \frac{1}{n!} = 0$.

b) $((-1)^n)_{n \in \mathbb{N}}$ alternates between -1 and 1, hence divergers

d) $(n)_{n \in \mathbb{N}}$ goes to infinity, so it does not converge to a real number

c) Using the definition of a limit:

For every $\varepsilon > 0$, we set $N = \frac{1}{\varepsilon}$. Then for all $n \geq N$,

$$\left| \frac{(-1)^n}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} = \varepsilon.$$

Here, I chose $N = \frac{1}{\varepsilon}$ after writing the second line...

I chose it so the distance with the limit would be at most ε.

General Technique

- Get an idea of what the limit should be.
- Prove it with the definition, or use the limit laws for sequence (see appendix).

Question: For what values of x does $(x^n)_{n \in \mathbb{N}}$ converge?

Make some examples, and come up with a conjecture.

Solution

The sequence $(x^n)_{n \in \mathbb{N}}$ is convergent if $-1 < x \leq 1$, and divergent otherwise. Moreover,

$$\lim_{n \to \infty} x^n = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } -1 < x < 1. \end{cases}$$

Question: What about $\lim_{n \to \infty} \underbrace{\sum_{k=0}^{n} x^k}_{x^0 + x^1 + \dots + x^n}$?

- For what values of x are you sure it diverges?
- For what values of x are you sure it converges?

Theorem

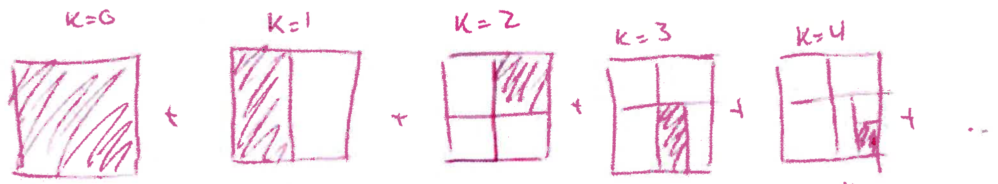
If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\left(\sum\limits_{k=0}^{n} a_k\right)_{n\in\mathbb{N}}$ diverges.

Solution to the question

- By the theorem, it diverges for $x \geq 1$ and $x \leq -1$.

Example: $x = 1/2$.

- For $r = 1/2$: we add the following



$\left(\frac{1}{2}\right)^0 = 1 \quad (1/2)^1 = 1/2 \quad (1/2)^2 = 1/4 \quad (1/2)^3 = 1/8 \quad (1/2)^4 = 1/16$

The sum is the full square

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This means $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} (1/2)^k = 2$.

Example: $x = 0$

$$\underbrace{0^0}_{1} + \underbrace{0^1}_{=0} + \underbrace{0^2}_{=0} + \underbrace{0^3}_{=0} + \dots = \underline{1}$$

Theorem

$\left(\sum\limits_{k=0}^{n} x^k\right)$ converges if $-1 < x < 1$, and diverges otherwise.

If $-1 < x < 1$,

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} x^k\right) = \frac{1}{1-x}$$

Note that $\sum_{k=0}^{n} x^k$ is the n-th Maclaurin polynomial of $\frac{1}{1-x}$. It is not a coincidence if when the sequence converges, it converges to this function.

Sketch of proof

Check that

$$(1-x)\lim_{n\to\infty}\left(\sum_{k=0}^{n} x^k\right)=1, \quad \text{when } -1<x<1.$$

Starting from the left-hand part,

$$\lim_{n\to\infty}(1-x)(1+x+x^2+\dots+x^n)=\lim_{n\to\infty}(1-x)+(x-x^2)+(x^2-x^3)+\dots+(x^n-x^{n+1})$$

$$=\lim_{n\to\infty} 1+(-x+x)+(-x^2+x^2)+\dots+(-x^n+x^n)-x^{n+1}$$

$$=\lim_{n\to\infty} 1-x^{n+1}$$

$$=1, \text{ when } -1<x<1, \text{ since } \lim_{n\to\infty} x^{n+1}=0.$$

$\boxtimes$

Question

What could the following be?

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}$$

Hint: it might be the limit of a Maclaurin polynomial.

Reference: Textbook, § 11.1 and 11.2.

Appendix: Example of a limit.

Question: Show that $\displaystyle\lim_{n\to\infty} \frac{3n^2+n+2}{n^2+1}$ is 3 using the definition of a limit.

Solution: We need to show that for all $\varepsilon>0$, there exists N (depending on ε) such that

$$\left| \frac{3n^2+n+2}{n^2+1} - 3 \right| \leq \varepsilon \quad \text{for all } n \geq N.$$

In other words, that the distance between the values of the sequence $\left(\frac{3n^2+n-2}{n^2+1}\right)_{n\in\mathbb{N}}$ and 3 gets arbitrarily close to 0.

To do so,

$$(\ast) = \left| \frac{3n^2+n+2}{n^2+1} - 3 \right| = \left| \frac{\cancel{3n^2}+n+2-\cancel{3n^2}-3}{n^2+1} \right| = \left| \frac{n-1}{n^2+1} \right|.$$

If $n \geq 1$, that is

$$\frac{n-1}{n^2+1} < \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}.$$

Since we want $(\ast)$ to be smaller than ε and we know it is smaller than $\frac{1}{n}$ (and hence, $\frac{1}{N}$, since $n \geq N$), we fix $N = \frac{1}{\varepsilon}$.

Then,

$$(\ast) < \frac{1}{n} < \frac{1}{N} = \varepsilon.$$