Math 8 - Lecture 11 Nadia Lafrenière

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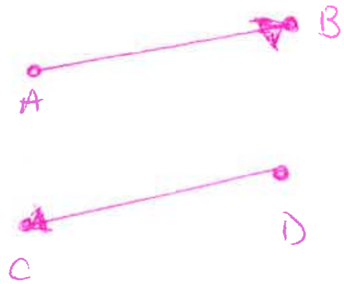
Vectors

The term vector is used by scientists from all fields to indicate a quantity for which both the magnitude and the direction are important.

The displacement vector $\vec{AB}$ is represented as an arrow on the line segment from A to B, where the tail of the vector is in A and its tip is at B. It corresponds to the displacement achieved by going from A to B.



A vector has two important data:
- its length
- its orientation.

Notice that the tail and the tip are not among this important data. This means that we only care about what the displacement achieved.

$\vec{AB} = -\vec{DC} = \vec{CD}$

For example, on the picture on the right, $\vec{AB} = \vec{CD}$.

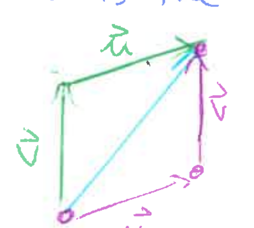
Two vectors that have the same length and orientation are equal or equivalent.

The zero vector corresponds to no displacement, and is denoted $\vec{0}$. It is the only vector that has no specific orientation.

Combining vectors

If two vectors $\vec{u}$ and $\vec{v}$ are placed such that the tip of $\vec{u}$ is the tail of $\vec{v}$, the sum of $\vec{u}$ and $\vec{v}$, denoted $\overrightarrow{u+v}$ is the vector from the tail of $\vec{u}$ to the tip of $\vec{v}$.

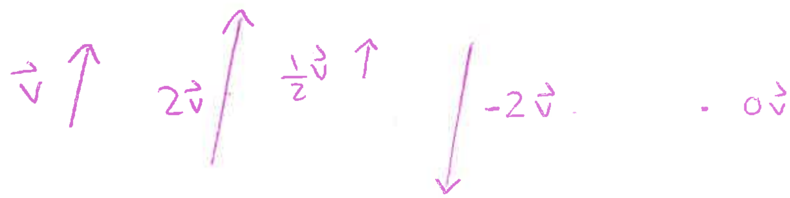
We can draw a parallelogram to see the following property: $\overrightarrow{u+v} = \overrightarrow{v+u}$



$\overrightarrow{u+v} = \overrightarrow{v+u}$

A scalar is a quantity that is not a vector, i.e. it is a (real) number.

If c is a scalar and $\vec{v}$ a vector, $c\vec{v}$ is a vector: its length is $|c|$ times the length of $\vec{v}$, and its orientation is the same as $\vec{v}$ if $c>0$, and opposite to the orientation of $\vec{v}$ if $c<0$.



The vector $-\vec{v}$ is the negative of $\vec{v}$.

The difference $\overrightarrow{u-v}$ is the sum of $\vec{u}$ and $-\vec{v}$.

The point of view I just presented is the geometric one. There is also an algebraic point of view:

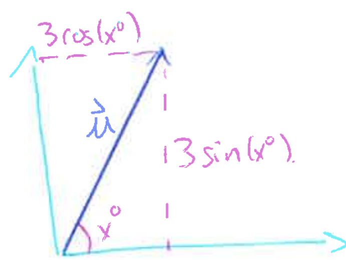
The vector $\vec{v}$ going from $A=(a_1,a_2,a_3)$ to $B=(b_1,b_2,b_3)$ is written using components as $\vec{v}=\langle b_1-a_1, b_2-a_2, b_3-a_3\rangle = \langle v_1, v_2, v_3\rangle$.

with angle brackets.

Its length is $\sqrt{v_1^2+v_2^2+v_3^2} =: |\vec{v}|$

If a vector $\vec{u}$ is two dimensional, has length 3 and has orientation $x°$ from the positive x-axis (counterclockwise), then $\vec{u}=\langle 3\cdot\cos(x°), 3\cdot\sin(x°)\rangle$

⚠ Caveat
$\vec{v}=\langle v_1, v_2, v_3\rangle$, and not $\langle\vec{v}_1,\vec{v}_2,\vec{v}_3\rangle$. The components are scalars.



Addition: $\overrightarrow{v+w} = \langle v_1+w_1, v_2+w_2, v_3+w_3\rangle$

Scalar multiplication: $c\cdot\vec{v} = \langle cv_1, cv_2, cv_3\rangle$.

Standard basis

There are three "special vectors" in $\mathbb{R}^3$.

$$\vec{i} = \langle 1,0,0 \rangle$$
$$\vec{j} = \langle 0,1,0 \rangle$$
$$\vec{k} = \langle 0,0,1 \rangle$$

This set of vectors is called the standard basis of $\mathbb{R}^3$. This means that all vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ can expressed as a weighted sum (called a linear combination) of $\vec{i}$, $\vec{j}$ and $\vec{k}$:

$$\vec{v} = v_1 \cdot \vec{i} + v_2 \cdot \vec{j} + v_3 \cdot \vec{k}.$$

These three vectors also have the property that they are unit vectors:

A unit vector is a vector of length 1. Except for the zero vector, all vectors $\vec{v}$ can be transformed into a unique unit vector that has the same orientation:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}.$$

Example

The unit vector associated to $\langle 2,-1,-2 \rangle$ is

$$\frac{1}{\sqrt{2^2+1^2+2^2}} \langle 2,-1,-2 \rangle = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \right\rangle.$$

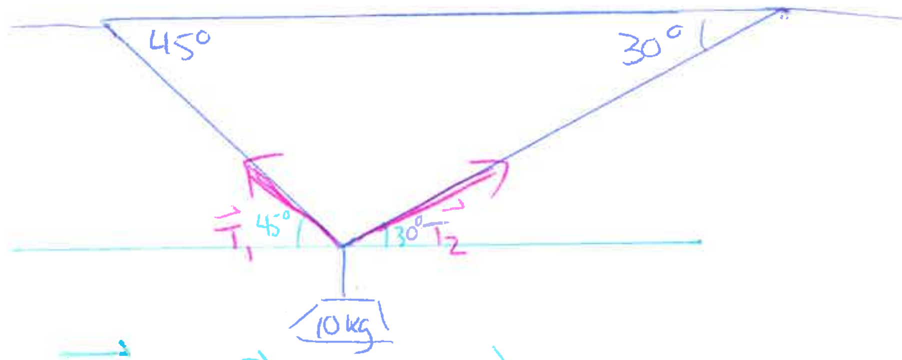
Application: The resultant force.

Vectors are also used to represent forces: they have a magnitude and a direction.

If there are many forces in a system, the resultant force is the vector sum of these forces.

Example

A 10 kg weight hangs from two wires as shown in the picture below. What are the magnitude of the tensions T_1 and T_2?



We know that $\overrightarrow{T_1+T_2} = -\vec{F_g}$, where $\vec{F_g}$ is the force caused by the gravity.

We also know that $\vec{F_g} = -10 \cdot 9.8\,\vec{j}$, where $\vec{j}$ is worth 1 N upward

So, $\overrightarrow{T_1+T_2} = 98\,\vec{j}$.

We also know that $\vec{T_1} = t_1 \underbrace{\langle \cos(135°), \sin(135°) \rangle}_{\text{unit vector}}$ and $\vec{T_2} = t_2 \langle \cos(30°), \sin(30°) \rangle$, where t_1 and t_2 are their respective magnitude

Thus,

$$\vec{T_1} = t_1 \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle \text{ and } \vec{T_2} = t_2 \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

We have the following system

$$0\vec{i} = \left(t_1 \left(-\frac{\sqrt{2}}{2}\right) + t_2 \frac{\sqrt{3}}{2}\right)\vec{i}$$

and

$$98\vec{j} = \left(t_1 \frac{\sqrt{2}}{2} + t_2/2\right)\vec{j}.$$

Therefore,

$$t_1 = \frac{\sqrt{3}}{\sqrt{2}} t_2 \quad \text{and} \quad 196 = \sqrt{2}\left(\frac{\sqrt{3}}{\sqrt{2}} t_2\right) + t_2 = (1+\sqrt{3})\, t_2,$$

and

$$t_2 = \frac{196}{1+\sqrt{3}} = 98(\sqrt{3}-1), \qquad t_1 = \frac{\sqrt{3}}{\sqrt{2}}\left(\frac{196}{1+\sqrt{3}}\right) = \frac{(3-\sqrt{3})\,98}{\sqrt{2}}.$$

Therefore, the magnitude of the tensions are

$$t_1 = \frac{(3-\sqrt{3})\,98}{\sqrt{2}}\ N \quad \text{and} \quad t_2 = 98(\sqrt{3}-1)\ N.$$

Extra problems (if time permits).

If the vectors below satisfy $|\vec{u}| = |\vec{v}| = 1$ and $\vec{u}+\vec{v}+\vec{w} = \vec{0}$, what is $\vec{w}$?

Find a unit vector that has the same orientation as $-5\vec{i}+3\vec{j}-\vec{k}$.

Reference: James STEWART, Calculus, 8^{th} edition, section 12.2.