

Dot product and projection

Problem: A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 30° above the horizontal. Find the work done by the force.

The problem is that the force is not applied in the same direction as the displacement.



Intermediate Step: What is the proportion of the force that is used for the displacement?

Orientation of the displacement: \rightarrow

Orientation of the force: \nearrow_{30°

The force is $\langle 70 \cdot \cos(30^\circ), 70 \cdot \sin(30^\circ) \rangle$ N, and only the first component counts for the work.

So, the work is

$$100 \cdot 70 \cdot \cos(30^\circ) \approx 6062 \text{ J.}$$

Here, we implicitly used the dot product, without knowing it.

Definition

If \vec{u} and \vec{v} are two vectors, their dot product is the real number defined as

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cdot \cos(\theta),$$

notation for
dot product \uparrow

Where θ is the angle between \vec{u} and \vec{v} .

In the problem earlier, we computed the dot product of the displacement vector and the force vector.

Example

Compute the following dot products:

$$(a) \langle -3, 5 \rangle \cdot \langle 5, 3 \rangle$$

$$(b) \langle -3, 5 \rangle \cdot \langle -3, 5 \rangle$$

Solution.

(a) In the first case, the two vectors are orthogonal (i.e. perpendicular to each other), and $\cos\left(\frac{\pi}{2}\right) = 0$.

(b) Here, we have the same vector twice, and the angle between them is 0. Hence,

$$|\langle -3, 5 \rangle|^2 \cos(0) = |\langle -3, 5 \rangle|^2 = 34$$

Properties of the dot product

(a) $\vec{u} \cdot \vec{v} = 0$ if and only if \vec{u} and \vec{v} are orthogonal
($\vec{0}$ is orthogonal to all vectors)

$$(b) \vec{u} \cdot \vec{u} = |\vec{u}|^2$$

$$(c) c \cdot \vec{u} \cdot \vec{v} = (c \cdot \vec{u}) \cdot \vec{v} = \vec{u} \cdot (c \cdot \vec{v}), \text{ for } c \in \mathbb{R}$$

$$\triangle! c \vec{u} \cdot \vec{v} \neq (c \cdot \vec{u}) \cdot (c \cdot \vec{v})$$

It is equivalent to multiplying the length of one vector.

$$(d) \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

We will talk about it once we know the algebraic notation.

Algebraic notation

Proposition

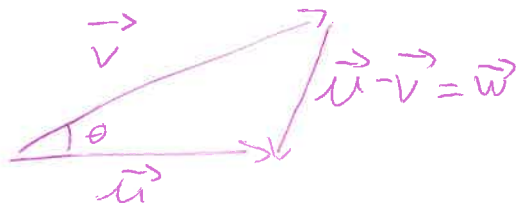
If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ are vectors, then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Proof

(3)

We have two 3-dimensional vectors \vec{u} and \vec{v} , so there exists a plane (i.e. a 2-dimensional object), that contains both, so we can draw them like this:



This is a triangle, so we can apply the Law of Cosines:

$$|\vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta)$$

$\underbrace{\vec{u} \cdot \vec{v}}$

Hence,

$$\begin{aligned} 2\vec{u} \cdot \vec{v} &= |\vec{u}|^2 + |\vec{v}|^2 - |\vec{u} - \vec{v}|^2 \\ &= (u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - ((u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2) \\ &= 2u_1v_1 + 2u_2v_2 + 2u_3v_3. \end{aligned}$$

thus,

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3. \quad \square$$

Using the latter definition of $\vec{u} \cdot \vec{v}$, it should be easy to understand why property (d) is true.

Problem: Find the angle between the vectors $\vec{a} = \langle 2, 2, -1 \rangle$ and

$$\vec{b} = \langle 5, -3, 2 \rangle.$$

Solution: On the one hand, we have

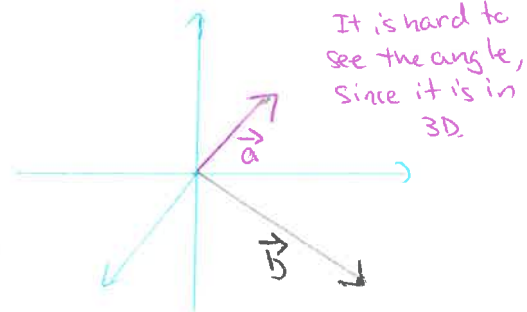
$$\vec{a} \cdot \vec{b} = 2 \cdot 5 + 2 \cdot (-3) + (-1) \cdot 2 = 2.$$

On the other hand,

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\theta) = 3 \cdot \sqrt{38} \cdot \cos(\theta),$$

so

$$\theta = \arccos\left(\frac{2}{3\sqrt{38}}\right) = 83.79^\circ$$



Projection

(4)

The dot product is the key to find the projections of a vector onto another.

Definition

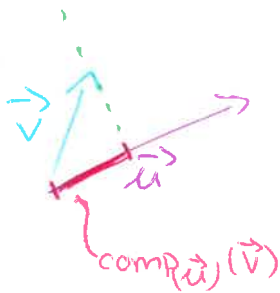
Given two vectors \vec{u} and \vec{v} , the vector projection of \vec{v} onto \vec{u} is the part of \vec{v} in the orientation of \vec{u} , and is given by

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2} \vec{u}$$

The scalar projection of \vec{v} onto \vec{u} is the signed magnitude of the projection (i.e. \pm the length). It is also called the component along \vec{v} , and is

$$\text{comp}_{\vec{u}}(\vec{v}) = \pm |\text{proj}_{\vec{u}}(\vec{v})| = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} = |\vec{v}| \cdot \cos(\theta)$$

where θ is the angle between \vec{u} and \vec{v} .



is the vector projection. Its length is $|\text{comp}_{\vec{u}}(\vec{v})|$.



Example

Find the vector and scalar projections of $\langle 1, 1, 2 \rangle$ onto $\langle -2, 3, 1 \rangle$

$$\langle 1, 1, 2 \rangle \cdot \langle -2, 3, 1 \rangle = 3, \quad |\langle -2, 3, 1 \rangle| = \sqrt{14}$$

Then,

$$\text{comp}_{\langle -2, 3, 1 \rangle}(\langle 1, 1, 2 \rangle) = \frac{3}{\sqrt{14}} \quad \text{and} \quad \text{proj}_{\langle -2, 3, 1 \rangle}(\langle 1, 1, 2 \rangle) = \frac{3}{14} \langle -2, 3, 1 \rangle = \left\langle \frac{-3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

Reference: James Stewart. Calculus, eighth edition. § 12.3.