Math 8 - Lecture 13 Nadia Lafrenière

Cross product. 02/03/2020

Problem

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors

How do you know if there exists scalars (i.e. real numbers) k, ℓ so that $k\vec{u} + \ell\vec{v} = \vec{w}$?

Examples for the 3-dim. vectors: • $\vec{u} = \langle -1,3,2 \rangle$, $\vec{v} = \langle 0,5,8 \rangle$, $\vec{w} = \langle 3,3,3 \rangle$ No
• $\vec{u} = \langle 1,4,-7 \rangle$, $\vec{v} = \langle 2,-1,4 \rangle$, $\vec{w} = \langle 0,-1,2 \rangle$: $k = -2/9$, $\ell = 1/9$

- If those vectors are two-dimensional and $\vec{u}$ and $\vec{v}$ are not colinear (i.e. parallel), any vector $\vec{w}$ can be written in that way. Said otherwise, $\vec{u}$ and $\vec{v}$ generate the whole plane.
- If two 3-dimensional vectors are not colinear, they generate a plane, so it depends on the fact that $\vec{w}$ is in that plane. (Give the examples above.)

How to do this?

We build a parallelepiped (a 3D analogue of a parallelogram), whose sides are $\vec{u}, \vec{v}$ and $\vec{w}$. If $\vec{u}, \vec{v}$ and $\vec{w}$ are in the same plane, the solid is flat and its volume is $\vec{0}$.
The volume of this solid is computed by the determinant of $\vec{u}, \vec{v}$ and $\vec{w}$

Definition (2D version)

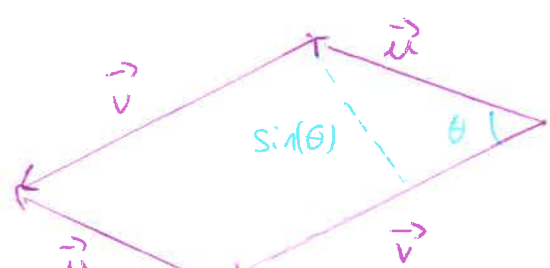
Let $\vec{u} = \langle a,b \rangle$ and $\vec{v} = \langle c,d \rangle$, (2-dimensional vectors)
The determinant of $\vec{u}$ and $\vec{v}$ is the (signed) area of the parallelogram generated by $\vec{u}$ and $\vec{v}$. It is positive if going from $\vec{u}$ to $\vec{v}$ creates an angle between 0° and 180° counterclockwise.
It is computed using

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = |\vec{u}||\vec{v}|\sin(\theta)$$

Example: $\vec{u} = \langle -3,1 \rangle$, $\vec{v} = \langle -4,-2 \rangle$

The area is

$$\begin{vmatrix} -3 & 1 \\ -4 & -2 \end{vmatrix} = (-3)(-2) - (-4)(1) = 10 \text{ units}^2$$



Definition

Let $\vec{u} = \langle a,b,c \rangle$, $\vec{v} = \langle d,e,f \rangle$, $\vec{w} = \langle g,h,i \rangle$ be 3-dimensional vectors. The determinant of $\vec{u}, \vec{v}$ and $\vec{w}$ is the (signed) volume of the parallelepiped generated by $\vec{u}, \vec{v}$ and $\vec{w}$, and is computed by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

$$= a(ei-hf) - b(di-gf) + c(dh-ge)$$

Tip to remember the formula:

- The numbers that multiply the 2D determinant are the numbers in the first row: $a, -b, c$, with the signs alternating
- The determinant that is multiplied by a is that obtained by deleting the row and column that contain a. It is the same thing for b and c.

The orientation is given by the right-hand rule (we will talk about it soon).

Example

- The vectors $\langle -1,3,2 \rangle$, $\langle 0,5,8 \rangle$, $\langle 3,3,3 \rangle$ have determinant

$$\begin{vmatrix} -1 & 3 & 2 \\ 0 & 5 & 8 \\ 3 & 3 & 3 \end{vmatrix} = -1 \begin{vmatrix} 5 & 8 \\ 3 & 3 \end{vmatrix} - 3 \begin{vmatrix} 0 & 8 \\ 3 & 3 \end{vmatrix} + 2 \begin{vmatrix} 0 & 5 \\ 3 & 3 \end{vmatrix}$$

$$= (-1)(-9) - 3(-24) + 2(-15)$$

$$= 9 + 72 - 30 = 51$$

Hence, the three vectors do not lie in the same plane (otherwise their volume would be 0), and there is no solution to

$$k \cdot \langle -1,3,2 \rangle + l \langle 0,5,8 \rangle = \langle 3,3,3 \rangle, \quad k,l \in \mathbb{R}.$$

Example

The vectors $\vec{u} = \langle 1,4,-7 \rangle$, $\vec{v} = \langle 2,-1,4 \rangle$ and $\vec{w} = \langle 0,-1,2 \rangle$ satisfy

$$-\frac{2}{9}\vec{u} + \frac{1}{9}\vec{v} = \vec{w}.$$

Hence, the parallelepiped generated by them is flat, and has volume 0:

$$\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -1 & 2 \end{vmatrix} = 1 \begin{vmatrix} -1 & 4 \\ -1 & 2 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + (-7) \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix}$$

$$= 1 \cdot 2 - 4 \cdot 4 + -7(-2)$$

$$= 2 - 16 + 14 = 0.$$

Problem

Given two non-parallel vectors $\vec{u}$ and $\vec{v}$, can you find a non-zero vector $\vec{w}$ that is orthogonal (i.e. perpendicular) to both $\vec{u}$ and $\vec{v}$?

This is a vector that gives a non-zero volume for the parallelepiped generated by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Idea: We will compute the determinant of $\vec{u}$, $\vec{v}$ and a vector containing the vectors of the standard basis $(\vec{i}, \vec{j}, \vec{k})$, and the result will be "a scalar" with $\vec{i}, \vec{j}$ and $\vec{k}$ as coefficients (and is thus a vector)

Definition

If $\vec{u} = \langle a,b,c \rangle$ and $\vec{v} = \langle d,e,f \rangle$, the cross product of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} = (bf-ec)\vec{i} - (af-dc)\vec{j} + (ae-bd)\vec{k}$$

$$= \langle bf-ec, dc-af, ae-bd \rangle.$$

Theorem

The vector $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

Proof

Two vectors are perpendicular if and only if their dot product is 0:

$$\vec{u}\cdot(\vec{u}\times\vec{v}) = \langle a,b,c\rangle \cdot \langle bf-ec, dc-af, ae-bd\rangle$$

$$= \cancel{abf} - \cancel{aec} + \cancel{bdc} - \cancel{baf} + \cancel{cae} - \cancel{cbd}$$

$$= 0.$$

And the same is true for $\vec{v}\cdot(\vec{u}\times\vec{v})$.

Example

Find a vector orthogonal to $\langle 1,4,-7\rangle$ and $\langle 2,-1,4\rangle$.

We compute the cross product:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & -7 \\ 2 & -1 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} 4 & -7 \\ -1 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -7 \\ 2 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$$

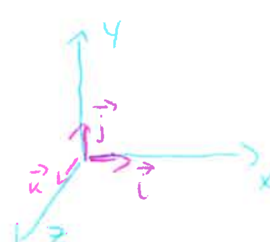
$$= \langle 9, -18, -9 \rangle$$

Orientation

The orientation for the cross product is given by the right hand rule: place your right hand and fold your fingers in the same orientation as the rotation from $\vec{u}$ to $\vec{v}$ (the shortest angle). Then your thumb indicates the orientation of $\vec{u}\times\vec{v}$.

Example

$\vec{k}\times\vec{j} = -\vec{i}$ and $\vec{j}\times\vec{k} = \vec{i}$



Properties of the cross product

(i) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

(ii) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
and
$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(iii) $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin(\theta)$, where θ is the shortest angle from $\vec{u}$ to $\vec{v}$

(iv) $|\vec{u} \times \vec{v}|$ is the area of the parallelogram generated by $\vec{u}$ and $\vec{v}$

(v) $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

Ideas for the proof

(i) By the right-hand rule.

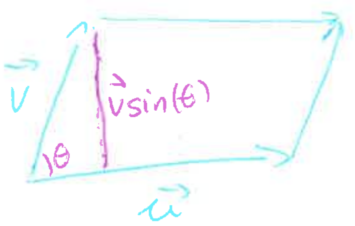
(ii) It is a straightforward computation.

(iii) Take $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle d, e, f \rangle$. Then

$$
\begin{aligned}
|\vec{u} \times \vec{v}|^2 &= \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} \right\|^2 \\
&= |\langle bf - ec, cd - af, ae - bd \rangle|^2 \\
&= (bf - ec)^2 + (cd - af)^2 + (ae - bd)^2 \\
&= b^2 f^2 + e^2 c^2 + c^2 d^2 + a^2 f^2 + a^2 e^2 + b^2 d^2 - 2bcef - 2acdf - 2abde \\
&= (a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - a^2 d^2 - b^2 e^2 - c^2 f^2 - 2bcef - 2acdf - 2abde \\
&= (a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - (ad + be + cf)^2 \\
&= |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 && \text{by algebraic definition of dot product.} \\
&= |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2 \theta) && \text{alternative definition of dot product} \\
&= |\vec{u}|^2 |\vec{v}|^2 (1 - \cos^2(\theta)) && \text{put } |\vec{u}|^2 |\vec{v}|^2 \text{ in evidence} \\
&= |\vec{u}|^2 \cdot |\vec{v}|^2 \sin^2(\theta) && \text{since } 1 = \sin^2\theta + \cos^2\theta.
\end{aligned}
$$

Then, $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin(\theta)$ by removing the square.

(iv) $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta)$ (by (iii)), so the area of the parallelogram below is $|\vec{u}||\vec{v}| \sin(\theta)$.



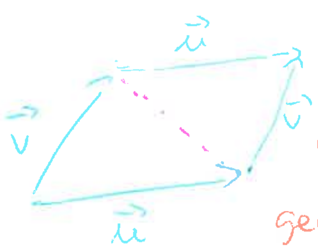
(v) If $\vec{u}$ and $\vec{v}$ are parallel, they generate a parallelogram of area 0.

has area 0.

Problems

(i) What is the area of the triangle with vertices (1,4,6), (-2,5,-1) and (1,-1,1)?

Solution: Find two vectors $\vec{u}$ and $\vec{v}$ from one vertex to another:



$$\vec{u} = (-2,5,-1) - (1,4,6) = \langle -3,1,-7 \rangle$$

and $\vec{v} = (1,4,6) - (1,-1,1) = \langle 0,5,5 \rangle$.

Use property (iv), so the area of the parallelogram generated by $\vec{u}$ and $\vec{v}$ is $|\vec{u} \times \vec{v}|$:

$$\left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & 5 & 5 \end{vmatrix} \right| = \left| \vec{i} \begin{vmatrix} 1 & -7 \\ 5 & 5 \end{vmatrix} - \vec{j} \begin{vmatrix} -3 & -7 \\ 0 & 5 \end{vmatrix} + \vec{k} \begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix} \right|$$

$$= \left| 40\vec{i} + 15\vec{j} - 15\vec{k} \right|$$

$$= \sqrt{40^2 + 15^2 + 15^2}$$

$$= 5\sqrt{8^2 + 3^2 + 3^2}$$

$$= 5\sqrt{82},$$

which is a little bit over 45 units^2.

(ii) Find a vector that is perpendicular to the plane containing $(1,4,6)$, $(-2,-5,-1)$ and $(1,-1,1)$.

This is the same plane as the one containing the triangle from the preceding example. Hence, the vector $\langle 40, 15, -15\rangle$ is orthogonal to it.

References: James STEWART. Calculus, 8^{th} edition, § 12.4.