

The goal of this lecture is to give the equations of lines and planes to better identify intersections and distances.

The necessary data to define a line is either

- two points
- one point and one vector
- an equation with exactly one free variable. (this is x' in $y=2x+3$; y is a dependent variable).

Of course, from two points, one can get a vector, and the converse is also true. The harder part seems thus to find the equation.

Vector equation

Given a point $p=(p_1, p_2, p_3)$ and a non-zero vector $\vec{v}=\langle v_1, v_2, v_3 \rangle$, the vector equation of the ^{unique} line L passing through p in the direction of \vec{v} is

$$\vec{L}(t) = \vec{p} + t\vec{v} = \langle p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 \rangle.$$

where t ranges over the real numbers, and \vec{p} is the position vector $\langle p_1, p_2, p_3 \rangle$.

The parametric equations of L is

$$x = p_1 + v_1 t, \quad y = p_2 + v_2 t, \quad z = p_3 + v_3 t.$$

Example

Vector and parametric equations for the line passing through $(5, 1, 3)$ and parallel to $\vec{i} + 4\vec{j} - 2\vec{k}$.

Here, the vector equation is

$$\vec{L}(t) = \langle 5, 1, 3 \rangle + t \langle 1, 4, -2 \rangle = \langle 5+t, 1+4t, 3-2t \rangle$$

and parametric equations are

$$x = 5+t, \quad y = 1+4t, \quad z = 3-2t.$$

Can you give two more points on that line?

(2)

- Set $t=1$ and $t=2$:

$$\vec{L}(1) = \langle 6, 5, 1 \rangle$$

and this is a position vector of the point $(6, 5, 1)$, that is on the line.

$$\vec{L}(2) = \langle 7, 9, -1 \rangle, \text{ and } (7, 9, -1) \text{ is on the line.}$$

Example

Find the parametric equations for the line passing through the points $P(2, 4, -3)$ and $Q(3, -1, 1)$.

(1) we find the vector $\vec{PQ} = \langle 1, -5, 4 \rangle$.

(2) The equations are thus:

$$x = 2 + t, \quad y = 4 - 5t, \quad z = -3 + 4t.$$

What is the intersection with the xy -plane?

On the xy -plane, the z -coordinate vanishes: $0 = -3 + 4t \Rightarrow t = 3/4$.

Then,

$$x = 2 + 3/4 = 11/4 \quad \text{and} \quad y = 4 - 5 \cdot 3/4 = 1/4$$

and the intersection is the point $(11/4, 1/4, 0)$.

In general, the intersection of a line with a plane is:

- * a line (if the line is in the plane)
- * a point (if the line is not parallel to a line in the plane, i.e. in most cases!)
- * empty, if it is parallel to a line in the plane.

The intersection of two lines can be

- * the whole line (for a line with itself)
- * a point, if the two lines belong to the same plane.
- * empty, in most cases, if the lines are skew or if they are parallel.

Example

What is the intersection of L_1 and L_2 , whose parametric equations are

$$L_1: \quad x = 1+t, \quad y = -2+3t, \quad z = 4-t.$$

$$L_2: \quad x = 2s, \quad y = 3+s, \quad z = -3+4s.$$

- The lines are not parallel, because the corresponding direction vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel.
- What is their intersection? The point that satisfies

$$2s = 1+t, \quad (1) \quad 3+s = -2+3t, \quad (2) \quad 4-t = -3+4s, \quad (3)$$

if any.

$$(1) \text{ gives } t = 2s - 1.$$

Along with (3), we get $-3+4s = 4-t = 4-(2s-1)$, so $s = \frac{4}{3}$ and $t = \frac{5}{3}$.

Does that satisfy (2)?

$$3+s = \frac{13}{3}, \quad \text{and} \quad -2+3t = 3.$$

No, so the intersection is empty.

Planes

The data to define a plane is either:

- 3 points, not in the same line
- 2 vectors and 1 point
- an equation with two free variables.

We already know how to go from the first to the second and backwards, but what about the equation?

Another thing that is enough to identify a plane is

- a vector orthogonal to all the vectors in the plane, and a point in it

Definition

(4)

To any plane, one can associate a normal vector, that is a vector orthogonal (perpendicular) to all the vectors in the plane.

Given any two non-parallel vectors \vec{u} and \vec{v} in the plane, $\vec{u} \times \vec{v}$ is a normal vector.

Remark: It is not unique, but all the normal vectors to a given plane have the same direction (they can have a different magnitude).

Let $P = (p_1, p_2, p_3)$ be a point in the plane, and \vec{n} a normal vector.
Let $Q = (q_1, q_2, q_3)$ a point. It is in the plane if and only if

$$\underbrace{\langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle}_{\text{Vector from } P \text{ to } Q} \cdot \vec{n} = 0$$

The equation

$$\langle x - p_1, y - p_2, z - p_3 \rangle \cdot \vec{n} = 0$$

is the vector equation of the plane orthogonal to \vec{n} containing P .

Now let $\vec{n} = \langle a, b, c \rangle$. Then,

$$a(x - p_1) + b(y - p_2) + c(z - p_3) = 0$$

is the scalar equation of the same plane.

Notice that, once we know \vec{n} and P , $ap_1 + bp_2 + cp_3$ is just a number.

Hence,

$$ax + by + cz + d = 0$$

is the linear equation of the same plane, when $d = -(ap_1 + bp_2 + cp_3)$.

Example

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Find an equation of the plane that passes through $P=(1,3,2)$, $Q=(3,-1,6)$ and $R=(5,2,0)$

(i) We find two vectors in the plane:

$$\vec{PQ} = \langle 2, -4, 4 \rangle$$

$$\vec{PR} = \langle 4, -1, -2 \rangle$$

(ii) Their normal vector is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \vec{i} \begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -4 \\ 4 & -1 \end{vmatrix}$$

$$= \langle 12, 20, 14 \rangle$$

(iii) The vector equation of the plane is

$$\langle x-1, y-3, z-2 \rangle \cdot \langle 12, 20, 14 \rangle = 0$$

$\underbrace{\hspace{10em}}_{\vec{PW}, \text{ with } \vec{W}=(x,y,z)}$

(iv) the scalar equation is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

(v) The linear equation is

$$12x + 20y + 14z - 100 = 0.$$

Relative position of two planes **I plan to finish the lecture here**

• Two planes are parallel if their normal vectors are parallel.

• The angle between two planes is the angle between their normal vectors.

How do you compute the angle between two vectors? Using the dot product!

Example

Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$

For the linear equations, the coefficients are the coordinates of the normal vectors: $\vec{n}_1 = \langle 1, 1, 1 \rangle$ and $\vec{n}_2 = \langle 1, -2, 3 \rangle$.

They are not parallel so they cross each other (planes cannot be skew).

The angle between them is

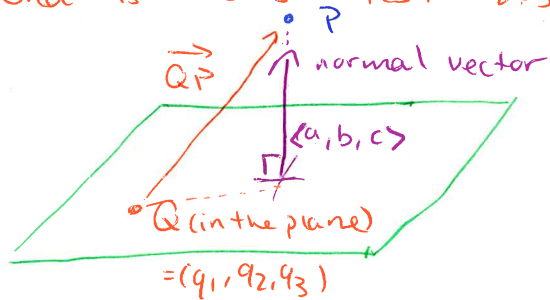
$$\arccos \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right) = \arccos \left(\frac{2}{\sqrt{3} \cdot \sqrt{14}} \right) = \arccos \left(\frac{2}{\sqrt{42}} \right) \approx 72^\circ.$$

Look on Geogebra? If time permits.

Distance

Problem: Consider a plane $ax+by+cz+d=0$, and a point $P=(p_1, p_2, p_3)$

What is the shortest distance D between P and the plane?



- (1) Choose a point in the plane, Q .
- (2) From the equation, we know that $\langle a, b, c \rangle$ is a normal vector.
- (3) We do not know anything about the magnitude of $\langle a, b, c \rangle$, but taking the scalar projection of \vec{QP} onto $\langle a, b, c \rangle$ would give the length of the part of \vec{QP} in the direction of the normal vector.

$$D = \left| \frac{\vec{QP} \cdot \langle a, b, c \rangle}{|\langle a, b, c \rangle|} \right|$$

This is the distance between P and the plane.

Problem: with the example from page 3, the lines are skew. What is the shortest distance between them?

Ideas on how to solve it:

- (1) take the direction vectors of L_1 and L_2
- (2) You get 2 vectors that are not parallel, so their cross product is a non-zero vector perpendicular to both lines.

(3) The latter vector can be seen as the normal vector to two parallel planes: one containing L_1 and the other one containing L_2 . The distance between these lines is thus the distance between the two planes

Exercise: find why the last sentence is true.

(4) The distance between two parallel planes is the distance between one point from the first ^{plane} and the second plane, so you can use the technique on the previous page.

The details of that problem can be found in the textbook (Example 10, §12.5).

Reference: James STEWART. Calculus, 8th edition, § 12.5.