Math 8 - Lecture 15 Nadia Lafrenière

02/07/2020

Vector functions and space curves

Last class, we introduced vector-valued functions to express lines: $\langle at+p_1, bt+p_2, ct+p_3 \rangle$.

We can do the same with non-linear functions as components.

Definition

Given three real-valued functions $f(t), g(t), h(t)$ (i.e. functions whose image is a real number),

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

is a vector-valued function.

How to think about this?

Think of t as time, and to the vector as the position vector of a particle at time t, where the x-coordinate is $f(t)$, ...

Example

$\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$ is a vector-valued function and is defined for all values of t for which all three of

$$\begin{cases} t^3 \\ \ln(3-t) \\ \sqrt{t} \end{cases}$$

is defined. That is $0 \leq t < 3$, the domain of $\vec{r}$.

Continuity

The limit of a vector-valued function is given by the limits of the components: if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$,

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle.$$

A vector-valued function $\vec{r}$ is continuous at a if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

Example

Find the limit as $t \to 0$ of $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \frac{\sin(t)}{t}\vec{k}$.

This is

$$\left(\lim_{t\to 0} (1+t^3)\right)\vec{i} + \left(\lim_{t\to 0} te^{-t}\right)\vec{j} + \left(\lim_{t\to 0} \frac{\sin(t)}{t}\right)\vec{k}$$

$$= \vec{i} + \vec{k}.$$

Space curves

Suppose $f(t)$, $g(t)$ and $h(t)$ are continuous function on an interval I. The set of images of $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ (seen as position vectors) as t varies throughout I is a space curve.

This is the set of points $(f(t), g(t), h(t))$ when $t \in I$.

Space curves and motion

If we see a vector-valued function $\vec{r}(t)$ as the position of a particle at time t, then a space curve is just the trajectory of the particle during an interval of time.

Example

Describe the curve defined by

$$\vec{r}(t) = \langle 1+t, 2+5t, -1+6t\rangle, \quad t \in \mathbb{R}.$$

As seen last lecture, this is just the line of direction vector $\langle 1,5,6\rangle$ and passing through $(1,2,-1)$.

This is also a curve.

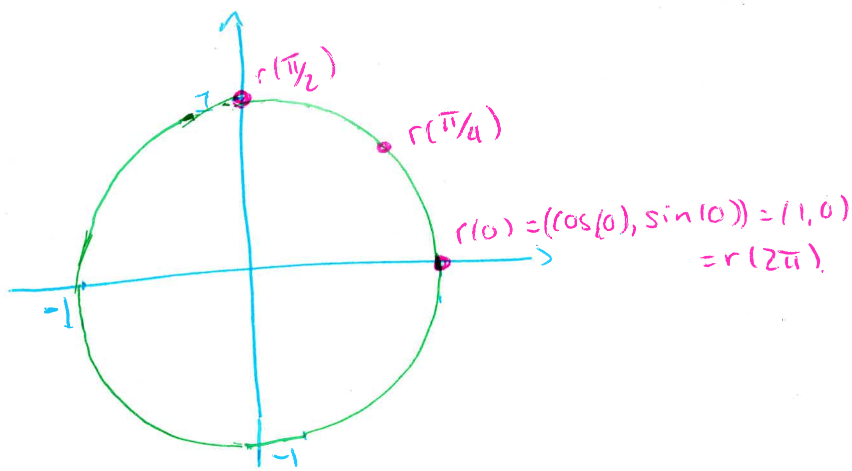
Example

Sketch the curve whose equation is

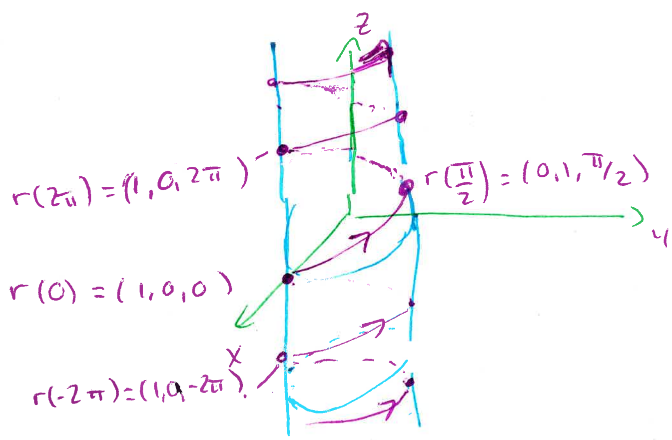
$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle.$$

(i) Look first at the projection on the xy-plane: $\langle \cos(t), \sin(t) \rangle$.

Drawing this curve for $t \in [0, 2\pi]$, I get



Hence the curve in three dimensions is at the surface of a cylinder (of radius 1, centered at (0,0,z)).



This curve is called the helix.

It is also possible to visualize it using geogebra!

Example

What is the vector equation of the line segment from point $P = (1,3,-2)$ to $Q = (2,-1,3)$?

We already know how to compute the vector $\vec{PQ}$ and give the equation of the line on which they lie.

But it could be simpler.

If we allow t to range from 0 to 1, we want to be in P at time $t=0$, Q at time $t=1$, and gradually move from P to Q.

This can be done with the equation

$$\vec{r}(t) = (1-t)\vec{P} + t\vec{Q}, \qquad 0 \le t \le 1,$$

so we do not even have to compute the vector.

Plugging the values of $\vec{P}$ and $\vec{Q}$, we get

$$\vec{r}(t) = (1-t)\langle 1,3,-2\rangle + t\langle 2,-1,3\rangle$$

$$= \langle 1-t+2t,\ 3-3t-t,\ -2+2t+3t\rangle$$

$$= \langle 1+t,\ 3-4t,\ -2+5t\rangle, \qquad 0 \le t \le 1.$$

Example

Can you find the curve for the intersection of the surfaces $x^2+y^2=1$ and $y+z=2$.

(i) Identify what type of surfaces these are.

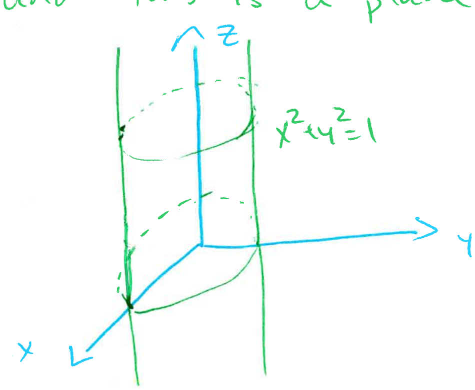
(ii) Sketch them, and sketch their intersection.

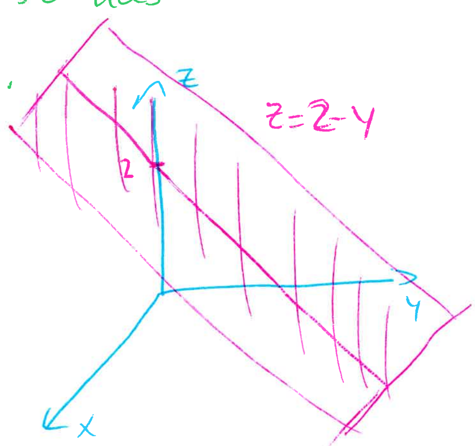
(iii) Compute the equation of the curve for the intersection.

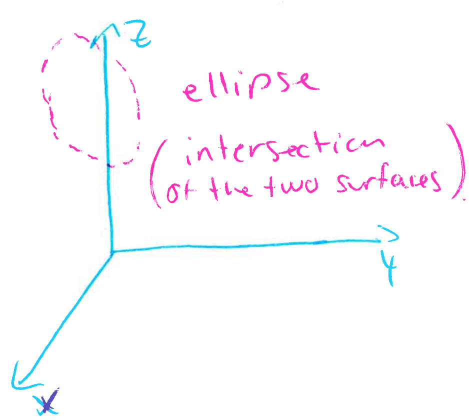
(i) As a 3D object, the one with equation $x^2+y^2=1$ is a surface (because it has 2 (including z) free variable). Moreover, for each value of t, the intersection of $x^2+y^2=1$ with the plane $z=t$ is a circle of radius 1 centered at $(0,0,t)$. So this surface is a cylinder.

The other equation also has two free variable (including x), and this is a plane.

(ii)







(iii) The first equation, since we get a circle, suggests that the two first coordinates would be $\cos(t)$ and $\sin(t)$, and that t would range from 0 to 2π.

For the parametric equations, we have

$$x = \cos t, \quad y = \sin t.$$

What is z? We know that $z = 2 - y = 2 - \sin t$.

Hence, the curve is defined by $\vec{r}(t) = \langle \cos(t), \sin(t), 2 - \sin(t) \rangle$.

Example

What is the vector equation of the ellipse centered at $(-3, 2)$ that has x-radius 5 and y-radius 1?

This is just a deformation of the circle centered at $(-3, 2)$. This circle has equation

$$\langle \cos t - 3, \sin t + 2 \rangle, \quad 0 \le t \le 2\pi.$$

However, this is a circle of radius 1. This is certainly correct for the y-radius here, but the x-radius should be 5, so we multiply in the first coordinate by 5 (sin t) to expand it:

$$\vec{r}(t) = \langle 5\cos(t) - 3, \sin(t) + 2 \rangle, \quad 0 \le t \le 2\pi$$

Reference: James STEWART. Calculus, §13.1.