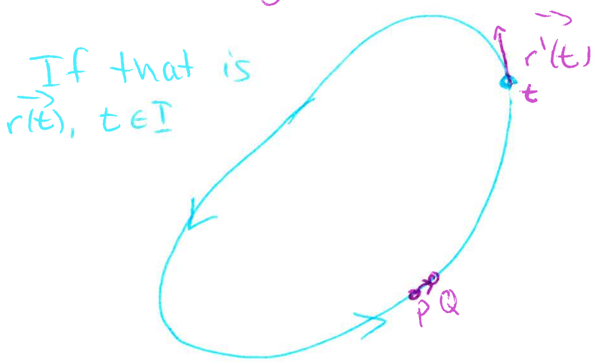
Math 8 - Lecture 16 Nadia Lafrenière

Derivatives and Integrals of vector-valued functions 02/10/2020

Last class, we introduced the vector valued functions as curves. How can we find the tangent vector to a curve $\vec{r(t)}$, at point $(f(t), g(t), h(t))$ (for a given t)?



Assume you have two points $P = (f(t_1), g(t_1), h(t_1))$ and $Q = (f(t_1+\Delta t), g(t_1+\Delta t), h(t_1+\Delta t))$ on the curve, the instantaneous variation rate is given by the vector $\frac{\vec{Q}-\vec{P}}{\Delta t}$, as $\Delta t \to 0$. (Here, $\vec{P}$ and $\vec{Q}$ are the position vectors of P and Q, i.e. $\vec{OP} = \vec{P}$ and $\vec{OQ} = \vec{Q}$)

$$\lim_{\Delta(t)\to 0} \left\langle \frac{f(t_1+\Delta t)}{\Delta t}, \frac{g(t_1+\Delta t)}{\Delta t}, \frac{h(t_1+\Delta t)}{\Delta t} \right\rangle$$

$$= \left\langle \lim_{\Delta t\to 0} \frac{f(t_1+\Delta t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t_1+\Delta t)}{\Delta t}, \lim_{\Delta t\to 0} \frac{h(t_1+\Delta t)}{\Delta t} \right\rangle$$

$$= \langle f'(t_1), g'(t_1), h'(t_1) \rangle.$$

The derivative of the vector-valued function $\vec{r(t)} = \langle f(t), g(t), h(t) \rangle$ is $\vec{r'(t)} = \lim_{\Delta t \to 0} \frac{\vec{r(t+\Delta t)} - \vec{r(t)}}{\Delta t}$, if this limit exists.

Equivalently, $\vec{r'(t)} = \langle f'(t), g'(t), h'(t) \rangle$, provided the three functions f, g and h are differentiable in t.

The tangent vector of the curve at point P such that $\vec{P} = \vec{r(t)}$ is $\vec{r'(t)}$ if that vector exists and is non-zero.

The tangent line is the line passing by P with direction vector $\vec{r'(t)}$.

If $\vec{r(t)}$ is the displacement of an object, then $\vec{r'(t)}$ is its velocity vector, and $|\vec{r'(t)}|$ is its speed.

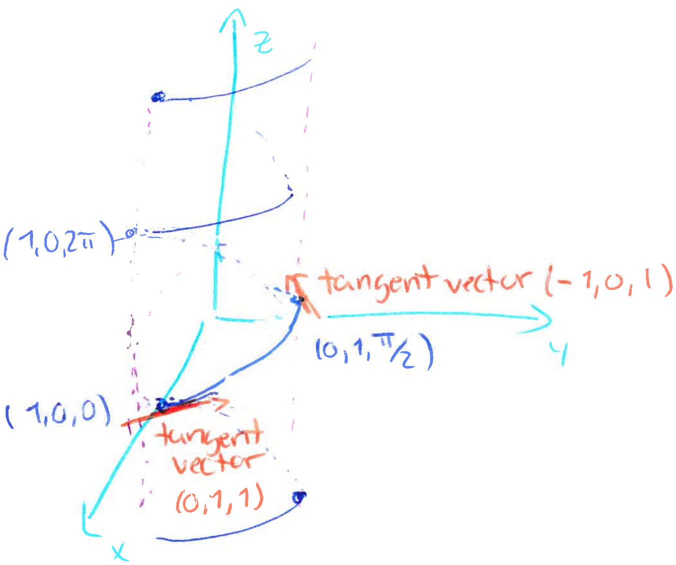
Example

Recall the equation of the helix:

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle, \quad t \in \mathbb{R}.$$

Its derivative is

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle, \quad t \in \mathbb{R}.$$



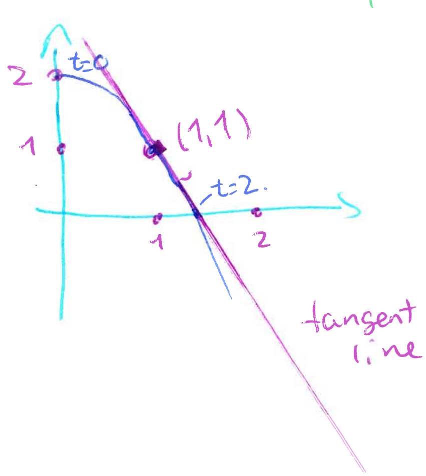
Can you sketch the tangent line passing through $(1,0,2\pi)$?

Example

Sketch the 2D-curve $\vec{r}(t) = \sqrt{t}\,\vec{i} + (2-t)\vec{j}$ and the tangent line passing through $(1,1)$. The domain of $\vec{r}(t)$ is $t \geq 0$.

(i) This curve is such that $x = \sqrt{t}$ and $y = 2-t = 2-x^2$.

So we can sketch $y = 2-x^2$.



(ii) $\vec{r}'(t) = \dfrac{1}{2\sqrt{t}}\vec{i} - t\,\vec{j}$

So $\vec{r}'(1) = \dfrac{1}{2}\vec{i} - \vec{j}$ is the direction vector of the tangent line passing through $(1,1)$.

(which implies that $y = -2x + C$, a constant that can be determined by $1 = -2\cdot 1 + C$. So $C = 3$, and $y = -2x + 3$)

The second derivative of a vector function $\vec{r}(t)$ is $\vec{r}''(t)$, the derivative of $\vec{r}'(t)$

It represents the acceleration vector.

Differentiation rules

Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions, f be a differentiable real-valued function and $c \in \mathbb{R}$. Then,

(i) $\frac{d}{dt}[\vec{u}(t)+\vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$ Addition.

(ii) $\frac{d}{dt}[c \cdot \vec{u}(t)] = c \cdot \vec{u}'(t)$ Scalar multiplication

(iii) $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$ Product rule

(iv) $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}(t)\cdot\vec{v}'(t) + \vec{u}'(t)\cdot\vec{v}(t)$ Dot product

(v) $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$ Cross product.

(vi) $\frac{d}{dt}[\overrightarrow{u(f(t))}] = \overrightarrow{u'(f(t))}\cdot f'(t)$ Chain rule

Example

Show that if $|\vec{r}(t)| = c$ (a constant), then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all t. (You can think of a sphere, since it is an object that verify this).

Since $|\vec{r}(t)| = c$, $\vec{r}(t)\cdot\vec{r}(t) = c^2$.

So, on the one hand, $\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)] = \frac{d}{dt}c^2 = 0$.

On the other hand, $\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)] = 2\vec{r}(t)\cdot\vec{r}'(t)$. (rule (iv)).

Hence, $\vec{r}(t)\cdot\overrightarrow{r'(t)} = 0$ for all t, and $\vec{r}'(t)$ is always orthogonal to $\vec{r}(t)$.

It would be great to have something like the Fundamental Theorem of Calculus for vector-valued functions; sort of an inverse to the derivative. 4

Integrals

The integral of a vector-valued function is the integral of its components. If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$$

To prove this, use the definition of an integral as the limit of a Riemann sum.

Example

Let $\vec{r}(t) = \langle -\sin(t), \cos(t), 1 \rangle$, $t \in \mathbb{R}$. (we saw earlier that this is the tangent vectors of the helix).

Then,

$$\int \vec{r}(t)\,dt = \left\langle -\int \sin(t)\,dt, \int \cos(t)\,dt, \int dt \right\rangle$$

$$= \langle \cos(t), \sin(t), t \rangle + \langle C_1, C_2, C_3 \rangle, \quad C_i \in \mathbb{R}.$$

Interpretation

Suppose we know the velocity of a function as t evolves, the integral of the velocity is the displacement vector.

Reference: James STEWART. Calculus, 8th edition. § 13.2.